ON THE CLASS $\mathcal{RSI}$ OF RATIONAL SCHUR FUNCTIONS INTERTWINING SOLUTIONS OF LINEAR DIFFERENTIAL EQUATIONS

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Abstract. In this paper we extend and solve in the class of functions $\mathcal{RSI}$ mentioned in the title, a number of problems originally set for the class $\mathcal{RS}$ of rational functions contractive in the open right-half plane, and unitary on the imaginary line with respect to some preassigned self-adjoint matrix. The problems we consider include the Schur algorithm, the partial realization problem and the Nevanlinna-Pick interpolation problem. The arguments rely on the one-to-one correspondence between elements in a given subclass of $\mathcal{RSI}$ and elements in $\mathcal{RS}$. Another important tool in the arguments is a new result pertaining to the classical tangential Schur algorithm.

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1991 *Mathematics Subject Classification.* Primary:47A48, 93C15, 93C35; Secondary: 16E45, 46E22.

Key words and phrases. Schur algorithm, time-invariant $2D$-systems, linear differential equations.

D. Alpay thanks the Earl Katz family for endowing the chair which supported his research. The research of the authors was supported in part by the Israel Science Foundation grant 1023/07.

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1. Introduction

Functions $S(\lambda)$, which are $\mathbb{C}^{p\times p}$-valued, analytic and contractive in the open right half plane $\mathbb{C}_+$, or equivalently, such that the kernel

$$K(\lambda, w) = \frac{I_p - S(w)^* S(\lambda)}{\lambda + w^*}$$

is positive in $\mathbb{C}_+$, play an important role in system theory, inverse scattering theory, network theory and related topics; see for instance [L], [BC], [DD], [He], [A]. Here, positivity of the kernel means that for every $n \in \mathbb{N}$ and every choice of points $w_1, \ldots, w_n \in \mathbb{C}_+$ and vectors $\xi_1, \ldots, \xi_n \in \mathbb{C}^{1\times p}$ the $n \times n$ Hermitian matrix

$$\left[\xi_i K(w_i, w_j)\xi_j^*)\right]_{i,j=1,\ldots n}$$

is positive (that is, has all its eigenvalues greater or equal to 0).

Far reaching generalizations of this class were introduced in [M, MVc], in the study of $2D$-linear systems (say, with respect to the variables (t_1, t_2)), invariant with respect to the variable t_1. To introduce the classes defined in these papers we first need a definition.

Definition 1.1. *Let σ_1, σ_2, γ and γ_* be $\mathbb{C}^{p\times p}$-valued functions, continuous on an interval $\mathrm{I} = [a, b]$. Suppose moreover that σ_1 and σ_2 take self-adjoint values, and that σ_1 is differentiable and invertible on I, and that the following relations hold:*

$$\gamma(t_2) + \gamma(t_2)^* = \gamma_*(t_2) + \gamma_*(t_2)^* = -\frac{d}{dt_2}\sigma_1(t_2), \quad t_2 \in \mathrm{I}.$$

Then $\sigma_1, \sigma_2, \gamma, \gamma_*$ and the interval I are called ***vessel parameters****.*

The class of functions $\boldsymbol{\mathcal{SI}}$ corresponding to some vessel parameters σ_1, σ_2, γ, γ_* and I was introduced in [M, MVc] (see Definition 2.6 below) and consists of the functions $S(\lambda, t_2)$ of two variables λ, t_2 such that for every $t_2 \in \mathrm{I}$ the function $S(\lambda, t_2)$ is meromorphic in $\mathbb{C}_+$ and the kernel

$$\frac{\sigma_1(t_2) - S(w, t_2)^* \sigma_1(t_2) S(\lambda, t_2)}{\lambda + w^*} \tag{1.1}$$

is positive for λ and ω in the domain of analyticity of $S(\lambda, t_2)$ in $\mathbb{C}_+$. For positive $\sigma_1(t_2)$, the positivity of the kernel implies that S is analytic in $\mathbb{C}_+$; see [Do], [A]. For general (invertible) $\sigma_1(t_2)$, the entries of S are of bounded type and S has only poles in $\mathbb{C}_+$; see [ADRS]. It is also required that $S(\lambda, t_2)$ is analytic at infinity for each t_2, with value I_p there, and that $S(\lambda, t_2)$ maps solutions of the *input* Linear Differential Equation (LDE) with the spectral parameter λ

$$\lambda\sigma_2(t_2)u(\lambda, t_2) - \sigma_1(t_2)\frac{\partial}{\partial t_2}u(\lambda, t_2) + \gamma(t_2)u(\lambda, t_2) = 0,$$

to solutions of the *output* LDE

$$\lambda\sigma_{2*}(t_2)y(\lambda, t_2) - \sigma_{1*}(t_2)\frac{\partial}{\partial t_2}y(\lambda, t_2) + \gamma_*(t_2)y(\lambda, t_2) = 0.$$

It is proved in [M, MVc] that elements of $\boldsymbol{\mathcal{SI}}$ are the transfer functions of t_1-invariant conservative $2D$ systems; see Section 2.

The purpose of this paper is to study various questions in $\boldsymbol{\mathcal{SI}}$ in the case of functions rational in λ. A key result is the following theorem, which we prove in the sequel; see Section 4.

Theorem 4.1 *Let us fix the parameters σ_1, σ_2, and γ, and the interval* I. *Then for every $t_2^0 \in \mathrm{I}$ there is a one-to-one correspondence between pairs (γ_*, S) such that $S \in \boldsymbol{\mathcal{SI}}$ and γ_* continuous in a neighborhood of t_2^0, and functions $Y(\lambda)$, meromorphic in $\mathbb{C}_+$, and with the following properties*

1. $Y(\infty) = I_p$,
2. $Y(\lambda)^*\sigma_1(t_2^0)Y(\lambda) \leq \sigma_1(t_2^0)$ *for* $\lambda \in \mathbb{C}_+$ *where* Y *is analytic, and*
3. $Y(\lambda)^*\sigma_1(t_2^0)Y(\lambda) = \sigma_1(t_2^0)$ *for every* λ *satisfying* $\Re\lambda = 0$, *and in a neighborhood of which* Y *is analytic.*

As mentioned above the $\sigma_1(t_2)$-contractivity of Y implies that Y is of bounded type in $\mathbb{C}_+$, and thus the asserted non-tangential limits exist almost everywhere. But it is important to note that the theorem does not consider these limits, but only the points λ in the domain of analyticity of Y.

Definition 1.2. *Functions Y with the properties in Theorem 4.1 will be called $\sigma_1(t_2^0)$-inner functions, and their class will be denoted by $\boldsymbol{\mathcal{S}}(t_2^0)$. The subclass of rational functions of $\boldsymbol{\mathcal{S}}(t_2^0)$ will be denoted by $\boldsymbol{\mathcal{RS}}(t_2^0)$.*

For the sequel, it is important to notice that the general tangential Schur algorithm developped in [AD] can be applied to functions in $\boldsymbol{\mathcal{S}}(t_2^0)$, and in particular in $\boldsymbol{\mathcal{RS}}$.

Definition 1.3. *$\mathcal{RSI}$ will denote the subclass of functions in $\mathcal{SI}$ which are rational in λ for every $t_2 \in \mathrm{I}$.*

The paper consists of five sections besides the introduction, and we now describe their content. In Section 2 we review the main results from [M] and [MVc] on t_1-invariant conservative $2D$-systems, relevant to the present work. In particular the class $\mathcal{SI}$ mentioned above consists of the transfer functions of these systems. In Section 3 we present the reproducing kernel space approach to the tangential Schur algorithm for the class $\mathcal{RS}$, as developped in [AD]. We obtain in particular new formulas which allow us to find the main operator in a realization of an element of $\mathcal{RS}$ after one iteration of the tangential Schur algorithm; see formulas (3.15), (3.17), (3.16) in Theorem 3.5. In Section 4 we develop the tangential Schur algorithm for a function $S(\lambda, t_2) \in \mathcal{RSI}$. Applying directly the theory of the previous section to $S(\lambda, t_2)$ leads to a new function which need not belong to $\mathcal{RSI}$. Instead, we apply the Schur algorithm to the $\sigma_1(t_2^0)$-inner function $S(\lambda, t_2^0)$ for some preassigned $t_2^0 \in \mathrm{I}$, and obtain a simpler (in terms of McMillan degree) $\sigma_1(t_2^0)$-inner function $S_0(\lambda, t_2^0)$. We use Theorem 4.1 to obtain an element in a class $\mathcal{RSI}$ from $S_0(\lambda, t_2^0)$. We call this procedure the tangential Schur algorithm for the class $\mathcal{RSI}$. We study in Section 5 the coefficients (called the *Markov moments*) $H_i(t_2)$ of the expansion of $S(\lambda, t_2)$ around $\lambda = \infty$

$$S(\lambda, t_2) = I_p - \sum_{i=0}^{\infty} \frac{1}{\lambda^{i+1}} H_i(t_2)$$

It turns out that the first Markov moment $H_0(t_2)$ satisfies the Lyapunov equation

(1.2)
$$\gamma_*(t_2) - \gamma(t_2) = \sigma_2(t_2) H_0(t_2) - \sigma_1(t_2) H_0(t_2) \sigma_1^{-1}(t_2) \sigma_2(t_2), \quad t_2 \in \mathrm{I},$$

which means that given the functions $\sigma_1, \sigma_2, \gamma$ and H_0 on I, one can uniquely reconstruct γ_*, and, as a result of Theorem 4.1, there will exists a unique function $S(\lambda, t_2)$ ($t_2 \in \mathrm{I}$) with the given Markov parameters. We solve the following problem:

Problem 1.4 (Partial realization)**.** *Suppose that we are given the functions σ_1, σ_2 and the Markov moments $H_i, i = 0, \ldots, n$ ($n \geq 0$) defined in* I*, and satisfying there the (necessary) conditions* (5.5)

$$H_{i+1}\sigma_1^{-1} + (-1)^{i+1} \sigma_1^{-1} H_{i+1}^* = \sum_{j=0}^{i} (-1)^{j+1} H_{i-j} \sigma_1^{-1} H_j^*$$

and (5.6)

$$(-1)^n H_{2n}\sigma_1^{-1} + \sum_{i=0}^{n-1} (-1)^{i+n}\sigma_1^{-1} H_i^* \sigma_1 H_{2n-1-i}\sigma_1^{-1} > 0,$$

for $i = 0, \ldots, n-1$. *Fix* $t_2^0 \in \mathrm{I}$. *Find all functions* γ *for which there exists* $S(\lambda, t_2) \in \boldsymbol{\mathcal{RSI}}$ *defined in a neighborhood of* t_2^0 *with the moments* $H_i(t_2)$ *for* γ_* *computed from* (1.2).

In Section 6 we study the Nevanlinna Pick interpolation problem in the class $\boldsymbol{\mathcal{RSI}}$. Let us recall that in classical Schur analysis, the solution of Nevanlinna Pick interpolation problem plays a special role; see for instance [FK], [Dy], [A]. Schur analysis gives a parametrization of all solutions (when they exist) for the given data. In the classical case the interpolation problem may be formulated in the following way:

Problem 1.5. *Suppose that we are given* N *pairs of points* $< w_j, s_j >$, $j = 1, \ldots, N$ *in* $\mathbb{D} \times \mathbb{D}$, *where* $\mathbb{D}$ *denotes the open unit disk. Then:*

1. *Give sufficient and necessary conditions, so that there exists a function* $s(z)$ *analytic and contractive in* $\mathbb{D}$, *and such that* $s(w_j) = s_j$, $j = 1, \ldots, N$.
2. *Describe the set of all solutions for this problem.*

As is well known, and originates with the work of Pick [P], Problem 1.5 is solvable if and only if the $N \times N$ Hermitian matrix

$$\mathbb{P} = [\frac{1 - s_\ell s_j^*}{1 - w_\ell w_j^*}] \geq 0.$$

Nevanlinna [N] then gave a parametrization of all solutions, i.e., solved the second part of Problem 1.5, in the form $S(\lambda) = T_\Theta(S_0(\lambda))$, where T_Θ denotes a linear fractional transformation uniquely determined from the initial data $< w_j, s_j >$, and where the parameter $S_0(\lambda)$ runs through all functions analytic and contractive in the open unit disk (that is, Schur functions). We refer to [BGR], [Dy] for more information on this classical topic.

As a generalization of Problem 1.5 in the class $\boldsymbol{\mathcal{RSI}}$ we consider the following question:

Problem 1.6 (Nevanlinna-Pick interpolation)**.** *Let* $\sigma_1, \sigma_2, \gamma, \gamma_*$ *be vessel parameters and let* I *be an interval. Let* $N \in \mathbb{N}$ *and* $w_i, i = 1, \ldots, N$ *be complex numbers. Suppose also that* N *input functions* $\xi_i(t_2)$ *satisfying* (2.16) *with corresponding spectral parameters* w_i*'s, and* N *output functions* $\eta_i(t_2)$ *satisfying* (2.17) *with the given* w_i*'s.*

(1) *Give sufficient and necessary conditions, so that there exists* $S(\lambda, t_2) \in \boldsymbol{\mathcal{RSI}}$ *such that* $S(w_j, t_2)\xi_i(t_2) = \eta_i(t_2)$, $j = 1, \dots, N$, *on a sub-interval of* I.
(2) *Describe the set of all solutions for this problem.*

One can also consider a harder problem, which can also be considered as a generalization of the classical Nevanlinna-Pick interpolation problem, which uses the fact that we can also specify the data for different values of t_2:

Problem 1.7. *Given* $\mathbb{C}^{p\times p}$*-valued functions* $\sigma_1, \sigma_2, \gamma$ *defined on* I*, and given* N *quadruples* $< t_2^j, w_j, \xi_j, \eta_j >$*, where* $t_2^j \in I, w_j \in \mathbb{C}_+$*,* $\xi_j, \eta_j \in \mathbb{C}^{1\times p}$ $j = 1, \dots N$*, then:*

(1) *Give sufficient and necessary conditions, so that there exists* γ_* *and* $S(\lambda, t_2) \in \boldsymbol{\mathcal{RSI}}$ *such that* $S(w_j, t_2^j)\xi_j = \eta_j$, $j = 1, \dots, N$, *on a sub-interval of* I *containing all the* t_2^j.
(2) *Describe the set of all solutions for this problem.*

If all the values $t_2^j = t_2^0$ are equal, we have to find a function $S(\lambda, t_2^0)$ satisfying $S(w_j, t_2^0)\xi_j = \eta_j$. Thus, the above problem is a generalization of the classical Nevanlinna-Pick interpolation problem. We also remark that we do not address the question of describing the set of all solutions.

Remarks: The present paper deals with the rational case; the general case will be treated in a forthcoming publication. Some of the results presented here have been announced in [AMV].

2. t_1 INVARIANT CONSERVATIVE $2D$ SYSTEMS.

The material in this section is taken from [M] and [MV1], where proofs and more details can be found. The origin of this theory can be found in the paper [Li].

2.1. **Definition.** An overdetermined conservative t_1-invariant $2D$ system is a linear input-state-output (i/s/o) system, which consists of operators depending only on the variable t_2 and is of the following form:

$$
(2.1)\quad I\Sigma : \begin{cases} \frac{\partial}{\partial t_1} x(t_1, t_2) = A_1(t_2)x(t_1, t_2) + \widetilde{B}_1(t_2)u(t_1, t_2) \\ x(t_1, t_2) = F(t_2, t_2^0)x(t_1, t_2^0) + \int\limits_{t_2^0}^{t_2} F(t_2, s)\widetilde{B}_2(s)u(t_1, s)ds \\ y(t_1, t_2) = u(t_1, t_2) - \widetilde{B}(t_2)^* x(t_1, t_2), \end{cases}
$$

where the variable t_1 belongs to $\mathbb{R}$, and the variable t_2 belongs to some interval I. Furthermore, the input $u(t_1, t_2)$ and the output $y(t_1, t_2)$ take

values in some Hilbert space $\mathcal{E}$ and the state $x(t_1,t_2)$ takes values in Hilbert spaces $\mathcal{H}_{t_2}$. We assume that $u(t_1,t_2)$ and $y(t_1,t_2)$ are continuous functions of each variable when the other variable is fixed. The operators of the system are supposed to satisfy the following:

Assumptions 2.1.

(1) *$A_1(t_2):\mathcal{H}_{t_2}\to\mathcal{H}_{t_2}$, and $\widetilde{B}(t_2):\mathcal{E}\to\mathcal{H}_{t_2}$ are bounded operators for all t_2,*
(2) *The functions $\sigma_1,\sigma_2,\gamma,\gamma_*:\mathcal{E}\to\mathcal{E}$, are continuous in the operator norm topology.*
(3) *$\sigma_1(t_2)$ is an invertible operator for every $t_2\in$ I.*
(4) *$F(t,s)$ is an evolution continuous semi-group.*

For continuous inputs $u(t_1,t_2)$, the inner state is continuously differentiable. Requiring now the invariance of the system transition from (t_1^0,t_2^0) to (t_1,t_2) via the points (t_1^0,t_2) and (t_1,t_2^0), is equivalent to the equality of second order partial derivatives of $x(t_1,t_2)$:

$$\frac{\partial^2}{\partial t_1\partial t_2}x(t_1,t_2)=\frac{\partial^2}{\partial t_2\partial t_1}x(t_1,t_2). \tag{2.2}$$

Substituting in this equality the system equations we obtain that for the free evolution $u(t_1,t_2)=0$ the so called Lax equation holds

$$A_1(t_2)=F(t_2,t_2^0)A_1(t_2^0)F(t_2^0,t_2). \tag{2.3}$$

Inserting (2.3) into (2.2) we see that the input $u(t_1,t_2)$ has to satisfy the following PDE

$$\begin{gathered}\widetilde{B}(t_2)\sigma_2(t_2)\tfrac{\partial}{\partial t_1}u(t_1,t_2)-\widetilde{B}(t_2)\sigma_1(t_2)\tfrac{\partial}{\partial t_2}u(t_1,t_2)-\\ \big(A_1(t_2)\widetilde{B}(t_2)\sigma_2(t_2)+F(t_2,t_2^0)\tfrac{\partial}{\partial t_2}[F(t_2^0,t_2)\widetilde{B}(t_2)\sigma_1(t_2)]\big)u(t_1,t_2)=0.\end{gathered}$$

Assuming the existence of a function $\gamma(t_2)$ satisfying

$$A_1(t_2)\widetilde{B}(t_2)\sigma_2(t_2)+F(t_2,t_2^0)\frac{\partial}{\partial s}[F(t_2^0,t_2)\widetilde{B}(t_2)\sigma_1(t_2)]=-\widetilde{B}(t_2)\gamma(t_2) \tag{2.4}$$

we obtain that it is enough that $u(t_1,t_2)$ satisfies the PDE

$$\sigma_2(t_2)\frac{\partial}{\partial t_1}u(t_1,t_2)-\sigma_1(t_2)\frac{\partial}{\partial t_2}u(t_1,t_2)+\gamma(t_2)u(t_1,t_2)=0. \tag{2.5}$$

The output $y(t_1,t_2)$ should satisfy the *output compatibility condition* of the same type as for the input compatibility condition (2.5), namely:

$$\sigma_2(t_2)\frac{\partial}{\partial t_1}y(t_1,t_2)-\sigma_1(t_2)\frac{\partial}{\partial t_2}y(t_1,t_2)+\gamma_*(t_2)y(t_1,t_2)=0. \tag{2.6}$$

Inserting here $y(t_1,t_2) = u(t_1,t_2) - \widetilde{B}(t_2)^* x(t_2,t_2)$ we obtain that

$$\begin{aligned} (2.7) \qquad 0 &= \sigma_2(t_2)\widetilde{B}(t_2)^* A_1(t_2) F(t_2,t_2^0) - \\ &-\sigma_1(t_2)\frac{\partial}{\partial t_2}[\widetilde{B}(t_2)^* F(t_2,t_2^0)] + \gamma_*(t_2)\widetilde{B}(t_2)^* F(t_2,t_2^0) \\ (2.8)\ \gamma(t_2) &= \sigma_1(t_2)\widetilde{B}(t_2)^*\widetilde{B}(t_2)\sigma_2(t_2) - \\ &-\sigma_2(t_2)\widetilde{B}(t_2)^*\widetilde{B}(t_2)\sigma_1(t_2) + \gamma_*(t_2). \end{aligned}$$

The fact that the system is lossless comes from the requirement of the so called *energy balance* equations:

$$\frac{\partial}{\partial t_i}\langle x(t_1,t_2), x(t_1,t_2)\rangle_{\mathcal{H}_{t_2}} + \langle \sigma_i(t_2) y(t_1,t_2), y(t_1,t_2)\rangle_{\mathcal{E}} = \\ = \langle \sigma_i(t_2) u(t_1,t_2), u(t_1,t_2)\rangle_{\mathcal{E}},\ i = 1,2,$$

which means that the energy of the output is distributed between the energy of the input and the change of the energy of the state of the system. Immediate consequences of this requirement are

$$\begin{aligned} (2.9) \qquad 0 &= A_1(t_2) + A_1^*(t_2) + \widetilde{B}(t_2)\sigma_1(t_2)\widetilde{B}(t_2)^*, \\ (2.10) \frac{d}{dt_2}[F^*(t_2,t_2^0)F(t_2,t_2^0)] &= F^*(t_2,t_2^0)\widetilde{B}(t_2)^*\sigma_2(t_2)\widetilde{B}(t_2)F(t_2,t_2^0). \end{aligned}$$

In this manner we obtain the notion of *conservative vessel in the integral form*, which is a collection of operators and spaces

$$\mathfrak{V} = (A_1(t_2), F(t_2,t_2^0), \widetilde{B}(t_2); \sigma_1(t_2), \sigma_2(t_2), \gamma(t_2), \gamma_*(t_2); \mathcal{H}_{t_2}, \mathcal{E})$$

where the operators satisfy the regularity assumptions 2.1, and the following vessel conditions:

$$0 = A_1(t_2) + A_1^*(t_2) + \widetilde{B}(t_2)^*\sigma_1(t_2)\widetilde{B}(t_2) \tag{2.9}$$

$$\|F(t_2,t_2^0)x(t_1,t_2^0)\|^2 - \|x(t_1,t_2^0)\|^2 = \\ = \int_{t_2^0}^{t_2}\langle \sigma_2(s)\widetilde{B}(s)x(t_1,s), \widetilde{B}(s)x(t_1,s)\rangle ds \tag{2.10}$$

$$F(t_2,t_2^0)A_1(t_2^0) = A_1(t_2)F(t_2,t_2^0) \tag{2.3}$$

$$0 = \tfrac{d}{dt_2}(F(t_2^0,t_2)\widetilde{B}(t_2)\sigma_1(t_2)) + \\ +F(t_2^0,t_2)A_1(t_2)\widetilde{B}(t_2)\sigma_2(t_2) + F(t_2^0,t_2)\widetilde{B}(t_2)\gamma(t_2) \tag{2.4}$$

$$0 = \sigma_1(t_2)\tfrac{\partial}{\partial t_2}[\widetilde{B}(t_2)^*F(t_2,t_2^0)] - \\ -\sigma_2(t_2)\widetilde{B}(t_2)^*A_1(t_2)F(t_2,t_2^0) - \gamma_*(t_2)\widetilde{B}(t_2)^*F(t_2,t_2^0) \tag{2.7}$$

$$\gamma(t_2) = -\sigma_2(t_2)\widetilde{B}(t_2)^*\widetilde{B}(t_2)\sigma_1(t_2) + \\ +\sigma_1(t_2)\widetilde{B}(t_2)^*\widetilde{B}(t_2)\sigma_2(t_2) + \gamma_*(t_2) \tag{2.8}$$

In order to simplify some notations we make the following definition

Definition 2.2. *Let*

$$\mathbf{U} = \mathbf{U}(\lambda_1, \lambda_2; t_2) = \sigma_2(t_2)\lambda_1 - \sigma_1(t_2)\lambda_2 + \gamma(t_2)$$

and similarly

$$\mathbf{U}_* = \mathbf{U}_*(\lambda_1, \lambda_2; t_2) = \sigma_2(t_2)\lambda_1 - \sigma_1(t_2)\lambda_2 + \gamma_*(t_2)$$

Then the vessel $\mathfrak{V}$ is naturally associated to the system (2.1)

$$\Sigma : \begin{cases} \frac{\partial}{\partial t_1} x(t_1, t_2) = A_1(t_2)\ x(t_1, t_2) + \widetilde{B}(t_2)\sigma_1(t_2)\ u(t_1, t_2) \\ x(t_1, t_2) = F(t_2, t_2^0)x(t_1, t_2^0) + \int\limits_{t_2^0}^{t_2} F(t_2, s)\widetilde{B}(s)\sigma_2(s)u(t_1, s)ds \\ y(t_1, t_2) = u(t_1, t_2) - \widetilde{B}(t_2)^*\ x(t_1, t_2). \end{cases}$$

with inputs and outputs satisfying the compatibility conditions (2.5) and (2.6), i.e. satisfy:

$$U(\frac{\partial}{\partial t_1}, \frac{\partial}{\partial t_2}; t_2)u(t_1, t_2) = 0, \quad U_*(\frac{\partial}{\partial t_1}, \frac{\partial}{\partial t_2}; t_2)y(t_1, t_2) = 0$$

The theory of such vessels, developed in [M, MV1] enables to find a more convenient form of the vessel. Denoting $\mathcal{H} = \mathcal{H}_{t_2^0}$, $A_1 = A_1(t_2^0)$, $F^*(t_2, t_2^0)F(t_2, t_2^0) = \mathbb{X}^{-1}(t_2)$ and $B(t_2) = F(t_2^0, t_2)\widetilde{B}(t_2)$, we shall obtain the following notion, first introduced in [M2].

Definition 2.3. *A (differential) conservative vessel associated to the vessel parameters is a collection of operators and spaces*

$$\mathfrak{V} = (A_1, B(t_2), \mathbb{X}(t_2); \sigma_1(t_2), \sigma_2(t_2), \gamma(t_2), \gamma_*(t_2); \mathcal{H}, \mathcal{E}), \tag{2.11}$$

where the operators satisfy the following vessel conditions:

$$0 = \frac{d}{dt_2}(B(t_2)\sigma_1(t_2)) + A_1 B(t_2)\sigma_2(t_2) + B(t_2)\gamma(t_2), \tag{2.12}$$

$$A_1\mathbb{X}(t_2) + \mathbb{X}(t_2)A_1^* = B(t_2)\sigma_1(t_2)B(t_2)^*, \tag{2.13}$$

$$\frac{d}{dt_2}\mathbb{X}(t_2) = B(t_2)\sigma_2(t_2)B(t_2)^*, \tag{2.14}$$

$$\begin{aligned} \gamma_*(t_2) = \gamma(t_2) + \sigma_1(t_2)B(t_2)^*\mathbb{X}^{-1}(t_2)B(t_2)\sigma_2(t_2)- \\ -\sigma_2(t_2)B(t_2)^*\mathbb{X}^{-1}(t_2)B(t_2)\sigma_1(t_2) \end{aligned} \tag{2.15}$$

This representation of a vessel is the most convenient when one focuses on the notion of transfer function, as we do in the next subsection.

2.2. **Transfer function.** Performing a separation of variables as follows

$$\begin{aligned} u(t_1,t_2) &= u_\lambda(t_2)e^{\lambda t_1},\\ x(t_1,t_2) &= x_\lambda(t_2)e^{\lambda t_1},\\ y(t_1,t_2) &= y_\lambda(t_2)e^{\lambda t_1}, \end{aligned}$$

we arrive at the notion of a transfer function. Note that $u(t_1,t_2)$ and $y(t_1,t_2)$ satisfy PDEs, but $u_\lambda(t_2)$ and $y_\lambda(t_2)$ are solutions of LDEs with spectral parameter λ,

$$\mathbf{U}(\lambda,\frac{\partial}{\partial t_2};t_2)u(t_1,t_2)=0, \tag{2.16}$$

$$\mathbf{U}(\lambda,\frac{\partial}{\partial t_2};t_2)y(t_1,t_2)=0. \tag{2.17}$$

The corresponding i/s/o system becomes

$$\begin{cases} x_\lambda(t_2)=(\lambda I-A_1(t_2))^{-1}B(t_2)\sigma_1(t_2)u_\lambda(t_2)\\ \frac{\partial}{\partial t_2}x_\lambda(t_2)=F(t_2,t_2^0)x_\lambda(t_2^0)+\int\limits_{t_2^0}^{t_2}F(t_2,s)\widetilde{B}_2(s)u_\lambda(s)ds\\ y_\lambda(t_2)=u_\lambda(t_2)-\widetilde{B}(t_2)^*x_\lambda(t_2) \end{cases}$$

The output $y_\lambda(t_2)=u_\lambda(t_2)-\widetilde{B}(t_2)^*x_\lambda(t_2)$ may be found from the first i/s/o equation:

$$y_\lambda(t_2)=S(\lambda,t_2)u_\lambda(t_2),$$

using the *transfer function*

$$\begin{aligned} S(\lambda,t_2) &= I-\widetilde{B}(t_2)^*(\lambda I-A_1(t_2))^{-1}\widetilde{B}(t_2)\sigma_1(t_2)\\ &= I-\widetilde{B}(t_2)^*(\lambda I-F(t_2,t_2^0)A_1(t_2^0)F(t_2^0,t_2))^{-1}\widetilde{B}(t_2)\sigma_1(t_2)\\ &= I-\widetilde{B}(t_2)^*F(t_2,t_2^0)(\lambda I-A_1(t_2^0))^{-1}F(t_2^0,t_2)\widetilde{B}(t_2)\sigma_1(t_2)\\ &= I-\widetilde{B}(t_2)^*F^*(t_2^0,t_2)F^*(t_2,t_2^0)F(t_2,t_2^0)(\lambda I-A_1)^{-1}B(t_2)\sigma_1(t_2)\\ &= I-B(t_2)^*\mathbb{X}^{-1}(t_2)(\lambda I-A_1)^{-1}B(t_2)\sigma_1(t_2) \end{aligned}$$

and we obtain that

$$S(\lambda,t_2)=I-B(t_2)^*\mathbb{X}^{-1}(t_2)(\lambda I-A_1)^{-1}B(t_2)\sigma_1(t_2). \tag{2.18}$$

Proposition 2.4. *The transfer function $S(\lambda,t_2)$ defined by* (2.18) *has the following properties:*

(1) *For all t_2, $S(\lambda,t_2)$ is an analytic function of λ in the neighborhood of ∞, where it satisfies:*

$$S(\infty,t_2)=I_p$$

(2) *For all λ, $S(\lambda,t_2)$ is a continuous function of t_2.*

(3) *For λ in the domain of analyticity of $S(\lambda,t_2)$:*

$$S(\lambda,t_2)^*\sigma_1(t_2)S(\lambda,t_2) \leq \sigma_1(t_2), \quad \Re\lambda > 0, \tag{2.19}$$

and

$$S(\lambda,t_2)^*\sigma_1(t_2)S(\lambda,t_2) = \sigma_1(t_2), \quad \Re\lambda = 0. \tag{2.20}$$

(4) *Maps solutions of the input LDE* (2.16) *with spectral parameter λ to the output LDE* (2.17) *with the same spectral parameter.*

Proof: These properties are easily checked, and follow from the definition of $S(\lambda,t_2)$:

$$S(\lambda,t_2) = I - \widetilde{B}(t_2)^*(\lambda I - A_1(t_2))^{-1}\widetilde{B}(t_2)\sigma_1(t_2).$$

The function $S(\lambda,t_2)$ is analytic for $\lambda > \|A_1(t_2)\|$ and since all the operators are bounded, we have $S(\infty,t_2) = I_p$. The second property follows from the regularity assumptions 2.1. The third property follows from straightforward calculations:

$$\begin{aligned}&S(\lambda,t_2)^*\sigma_1(t_2)S(\lambda,t_2) - \sigma_1(t_2) = \\ &-2\Re(\lambda)\sigma_1(t_2)\widetilde{B}(t_2)^*(\bar{\lambda}I - A_1^*(t_2))^{-1}(\lambda I - A_1(t_2))^{-1}\widetilde{B}(t_2)\sigma_1(t_2)\end{aligned}$$

Here the sign of $-\Re(\lambda)$ determines the sign of $S(\lambda,t_2)^*\sigma_1(t_2)S(\lambda,t_2) - \sigma_1(t_2)$ and thus the third property is obtained. The fourth property follows directly from our construction. □

Remark: When $\dim\mathcal{H} < \infty$, we obtain that $S(\lambda,t_2)$ is a rational function of λ for every t_2.

It is an interesting fact that also the converse of Proposition 2.4 holds. It is proved in [M], [MVc, chapter 5] .

Theorem 2.5. *For any function of two variables $S(\lambda,t_2)$, satisfying the conditions of Proposition 2.4, there is a conservative t_1-invariant vessel whose transfer function is $S(\lambda,t_2)$.*

We define the class of transfer functions mentioned in the introduction as follows:

Definition 2.6 ([MVc])**.** *The class $\boldsymbol{\mathcal{SI}} = \boldsymbol{\mathcal{SI}}(\mathbf{U}(\lambda,\frac{\partial}{\partial t_2};t_2),\mathbf{U}_*(\lambda,\frac{\partial}{\partial t_2};t_2))$ consists of functions $S(\lambda,t_2)$ of two variables, which are*

(1) *analytic in a neighborhood of $\lambda=\infty$ for all t_2 and where it holds $S(\infty,t_2)=I_p$,*
(2) *continuous as functions of t_2 for all λ,*
(3) *satisfy* (2.19) *and* (2.20) *in the domain of analyticity of S,*

(4) *map solutions of the input LDE (2.16) with spectral parameter λ to the output LDE (2.17) with the same spectral parameter*

Recall (see [CoLe]) that to every LDE can be associated an invertible matrix (or operator) function $\Phi(t_2, t_2^0)$, called the *fundamental solution*, which takes value I at some preassigned value t_2^0 and such that any other solution $u(t_2)$ of the LDE, with initial condition $u(t_2^0) = u_0$ is of the form

$$u(t_2) = \Phi(t_2, t_2^0)u_0.$$

Let $\Phi(\lambda, t_2, t_2^0)$ and $\Phi_*(\lambda, t_2, t_2^0)$ be the fundamental solutions of the input LDE (2.16) and the output LDE (2.17) respectively, where we have added in the notation the dependence in λ. Then,

$$S(\lambda, t_2)\Phi(\lambda, t_2, t_2^0) = \Phi_*(\lambda, t_2, t_2^0)S(\lambda, t_2^0) \tag{2.21}$$

and consequently $S(\lambda, t_2)$ satisfies the following LDE

$$\begin{aligned} &\tfrac{\partial}{\partial t_2}S(\lambda, t_2) = \sigma_1^{-1}(t_2)(\sigma_2(t_2)\lambda + \gamma_*(t_2))S(\lambda, t_2) - \\ &-S(\lambda, t_2)\sigma_1^{-1}(t_2)(\sigma_2(t_2)\lambda + \gamma(t_2)). \end{aligned} \tag{2.22}$$

3. Schur analysis for the classical case

One of the approaches to the tangential Schur algorithm for a rational matrix function $S(\lambda) \in \mathbb{C}^{p\times p}$ is based on the theory of reproducing kernel Hilbert spaces of the kind introduced by de Branges and Rovnyak; see [dBR1], [dBR2], [Dy] for information on these spaces. The paper [AD] considers the case of column-valued functions. In this section we adapt the results of [AD] to the case of row-valued functions. We also present a new formula for a realization of a Schur function after implementation of the tangential Schur algorithm.

3.1. **Schur functions and Reproducing Kernel Hilbert spaces.**

In this section σ_1 denotes a fixed self-adjoint and invertible (but not necessarily unitary) matrix in $\mathbb{C}^{p\times p}$. Let $S(\lambda)$ be a rational function, σ_1-inner in the open right half plane, i.e.

$$S(\lambda)^*\sigma_1 S(\lambda) - \sigma_1 \leq 0$$

at all points in the domain of analyticity $\Omega(S)$ of S in $\mathbb{C}_+$, and

$$S(\lambda)^*\sigma_1 S(\lambda) - \sigma_1 = 0$$

at all points on the imaginary axis where S is defined. Then, the kernel

$$K_S(\lambda, w) = \frac{\sigma_1 - S(w)^*\sigma_1 S(\lambda)}{\bar{w} + \lambda} \tag{3.1}$$

is positive for $\lambda, w \in \Omega(S)$, and the space of rational $\mathbb{C}^{1\times p}$-valued functions

$$\mathcal{H}(S) = \{\sum_{i=1}^{n} \alpha_i c_i K_S(\lambda, w_i) \mid \alpha_1 \in \mathbb{C}, w_i \in \Omega, c_i \in \mathbb{C}^{1\times p}\}$$

is finite dimensional. These well-known facts can be proved using realization theory; see for instance [AG]. Furthermore, $\mathcal{H}(S)$ is the reproducing kernel Hilbert space, associated to the kernel $K_S(\lambda, w)$. The inner product is defined by

$$\langle cK_S(\lambda,\nu), dK_S(\lambda,w)\rangle_{\mathcal{H}_S} = \langle cK_S(w,\nu)\rangle_{C^{1\times p}} = cK_S(w,\nu)d^*.$$

For an arbitrary $f(\lambda) \in \mathcal{H}(S)$ we have the reproducing kernel property

$$\langle f(\lambda), \xi K(\lambda, w)\rangle_{\mathcal{H}_S} = f(w)\xi^*.$$

More generally, let now $\mathcal{M}$ be a finite dimensional Hilbert space of $\mathbb{C}^{1\times p}$-valued functions defined in some set Ω, and let $\{f_1(\lambda), \ldots, f_N(\lambda)\}$ be a basis of $\mathcal{M}$. Let $\mathbb{X} \in \mathbb{C}^{p\times p}$ denote the Gram matrix with ℓ, j entry given by

$$\mathbb{X}_{\ell,j} = \langle f_j(\lambda), f_\ell(\lambda)\rangle_{\mathcal{M}}, \quad \ell, j = 1, \ldots, p. \tag{3.2}$$

It is easily seen that the space $\mathcal{M}$ is a reproducing kernel Hilbert space with kernel given by the formula

$$K(\lambda, w) = \begin{bmatrix} f_1(w)^* & \cdots & f_N(w)^* \end{bmatrix} \mathbb{X}^{-1} \begin{bmatrix} f_1(\lambda) \\ \vdots \\ f_N(\lambda) \end{bmatrix}. \tag{3.3}$$

We set

$$F(\lambda) = \begin{bmatrix} f_1(\lambda) \\ \vdots \\ f_p(\lambda) \end{bmatrix}. \tag{3.4}$$

Assume now that $\mathcal{M}$ consists of rational functions, defined on a set $\Omega(\mathcal{M})$. For $\alpha \in \Omega(\mathcal{M})$ the backward-shift operator R_α is defined by

$$R_\alpha(f(\lambda)) = \frac{f(\lambda) - f(\alpha)}{\lambda - \alpha}.$$

Suppose that:

(1) The space $\mathcal{M}$ is invariant under the action of R_α,
(2) The functions $f_i(\lambda)$ has the property that $f_i(\infty) = 0$, i.e., $F(\infty) = 0$,

then the function F given by (3.4) can be written as

$$F(\lambda) = (\lambda I - A)^{-1} B\sigma_1,$$

for suitably chosen matrices A, B. In this special case, formula (3.3) takes the form

$$K(\lambda, w) = \sigma_1 B^*(\bar{w}I - A^*)^{-1}\mathbb{X}^{-1}(\lambda I - A)^{-1}B\sigma_1. \tag{3.5}$$

As mentioned at the beginning of this section we are interested in kernels of the form (3.1) for some σ_1-inner rational function S. We now recall the characterization of these spaces, and first note the following: equation (3.1) leads to

$$\frac{\sigma_1 - S(w)^*\sigma_1 S(\lambda)}{\bar{w} + \lambda} = F(w)^*\mathbb{X}^{-1}F(\lambda). \tag{3.6}$$

If S is analytic at infinity and satisfies there $S(\infty) = I_p$, and letting $w \to \infty$ in this equation, we obtain the formula

$$S(\lambda) = I_p - B^*\mathbb{X}^{-1}(\lambda I - A)^{-1}B\sigma_1. \tag{3.7}$$

Theorem 3.1. *Let $\mathcal{M}$ be a finite dimensional Hilbert space of $\mathbb{C}^{1\times p}$-valued rational functions, which are zero at infinity. Suppose that $R_\alpha\mathcal{M} \subset \mathcal{M}$ for $\alpha \in \Omega(\mathcal{M})$ and let $\mathbb{X}$ be its Gram matrix with respect to $F(\lambda)$. Then $\mathcal{M} = \mathcal{H}(S)$ for S defined by (3.7) if and only if the Lyapunov equation*

$$A\mathbb{X} + \mathbb{X}A^* + B\sigma_1 B^* = 0 \tag{3.8}$$

holds.

When the spectrum of the operator A is in the open left half plane $\Re\lambda < 0$, one has

$$\mathcal{H}(S) = \mathbf{H}_{2,\sigma_1} \ominus \mathbf{H}_{2,\sigma_1}S,$$

where $\mathbf{H}_{2,\sigma_1}$ is the Hardy space $\mathbf{H}_2^p$ with the inner product

$$[f, g]_{\mathbf{H}_{2,\sigma_1}} = \langle f, g\sigma_1^{-1}\rangle_{\mathbf{H}_2^p}.$$

We set $J = \begin{bmatrix} -\sigma_1 & 0 \\ 0 & \sigma_1 \end{bmatrix}$. Note that J is both invertible and self-adjoint, and one can define J-inner rational functions. Let $\Theta = \begin{bmatrix} \Theta_{11} & \Theta_{12} \\ \Theta_{21} & \Theta_{22} \end{bmatrix}$ be a J-inner rational function; we introduce the linear fractional transformation

$$T_\Theta(W) = (\Theta_{11} + W\Theta_{21})^{-1}(\Theta_{12} + W\Theta_{22}).$$

Theorem 3.2. *Let S and Θ be respectively σ_1-inner J-inner rational functions. Then there exists a σ_1-inner rational function W such that $S = T_\Theta(W)$ if and only if the map*

$$F \mapsto F \begin{bmatrix} -S(\lambda) \\ I_p \end{bmatrix} \tag{3.9}$$

is a contraction from $\mathcal{H}(\Theta)$ to $\mathcal{H}(S)$.

This theorem originates with the work of de Branges and Rovnyak (see [dBR1, Theorem 13 p. 305]), where it is proved in a more general setting, and is one of the key ingredients to the reproducing kernel approach to the Schur algorithm. The proof is the same as for column-valued functions, and is omitted.

As a special case of the previous theorem we have:

Corollary 3.3. *Given $S \in \mathcal{RS}$, $w_1, \ldots, w_n \in \mathbb{C}_+$, and row vectors $\xi_1, \ldots, \xi_n \in \mathbb{C}^{1\times p}$, define*

$$B = \begin{bmatrix} -\xi_1 S(w_1)^* & \xi_1 \\ \vdots & \vdots \\ -\xi_n S(w_n)^* & \xi_n \end{bmatrix}, \ A_1 = \mathrm{diag}[-w_1^*, \ldots, -w_n^*].$$

Let $\mathcal{M}$ be a Hilbert space of row vectors spanned by the rows of the matrix-valued function

$$F(\lambda) = (\lambda I - A_1)^{-1} BJ.$$

Let $\mathbb{X}$ be the solution of the Lyapunov equation

$$A\mathbb{X} + \mathbb{X}A^* + BJB^* = 0 \tag{3.10}$$

and assume $\mathbb{X} > 0$. Let Θ be defined by

$$\Theta(\lambda) = I_{2p} - B^*\mathbb{X}^{-1}(\lambda I_n - A_1)^{-1}BJ. \tag{3.11}$$

Then there exists $S_0 \in \mathcal{RS}$ such that $S = T_\Theta(S_0)$.

Proof: Let us denote by t the map (3.9). For an arbitrary element $f(\lambda) = \eta F(\lambda)$, we shall obtain that

$$\begin{aligned} tf(\lambda) &= \eta(\lambda I - A_1)^{-1}BJ \begin{bmatrix} -S(\lambda) \\ I_p \end{bmatrix} = \\ &= \eta\,\mathrm{diag}[\tfrac{1}{\lambda+\bar{w}_i}] \begin{bmatrix} -\xi_1 S(w_1)^* & \xi_1 \\ \vdots & \vdots \\ -\xi_n S(w_n)^* & \xi_n \end{bmatrix} \begin{bmatrix} \sigma_1 S \\ \sigma_1 \end{bmatrix} = \\ &= \textstyle\sum_i \frac{\eta_i}{\lambda+\bar{w}_i}\xi_i(\sigma_1 - S(w_i)^*\sigma_1 S(\lambda)) = \\ &= \textstyle\sum_i \xi_i\eta_i K_S(\lambda, w_i) \end{aligned}$$

and consequently,

$$\begin{aligned}\langle tf, tf\rangle &= \langle \textstyle\sum_i \xi_i\eta_i K_S(\lambda, w_i), \sum_j \xi_j\eta_i K_S(\lambda, w_j)\rangle = \\ &= \textstyle\sum_{ij} \eta_i\xi_i K_S(w_j, w_i)\eta_j^*\xi_j^* = \eta\mathbb{X}\eta^* = \\ &= \langle f, f\rangle.\end{aligned}$$

Thus t is an isometry and Theorem 3.2 allows to conclude. □

The following result shows that the assumption $\mathbb{X} > 0$ in the statement of Corollary 3.3 can always be achieved for $n = 1$.

Lemma 3.4. *Given $S \in \boldsymbol{\mathcal{RS}}$ which is not the function identically equal to I_p. Then there exist a pair $(\xi, w) \in \mathbb{C}^{1\times p} \times \mathbb{C}_+$ such that the corresponding $\mathbb{X} > 0$.*

Proof: We proceed by contradiction. Assume that for each $w \in \mathbb{C}_+$ and for each vector ξ, it holds that

$$\xi\sigma_1\xi^* = \xi S(w)^*\sigma_1 S(w)\xi^*.$$

or, equivalently,

$$\xi K_S(w, w)\xi^* = 0.$$

Then, $\xi K_S(\lambda, w) \equiv 0$, and for each $f \in \mathcal{H}(S)$

$$\xi f(w) =< f(\lambda, \xi K_S(\lambda, w) >= 0.$$

The space $\mathcal{H}(S)$ is thus trivial, and its kernel is zero:

$$\frac{\sigma_1 - S(w)^*\sigma_1 S(\lambda)}{\lambda + \bar{w}} = 0,$$

from where we conclude that for each w, λ

$$S(\lambda) = \sigma_1^{-1} S^{-*}(w)\sigma_1.$$

and consequently, $S(\lambda) \equiv I_p$. □

3.2. **Analysis of the tangential Schur algorithm.** For $n = 1$, the matrix function in (3.11) becomes

(3.12)

$$\begin{aligned}\Theta(\lambda) &= I_{2p} - B^*\mathbb{X}^{-1}(\lambda I - A)^{-1}BJ = \\ &= I_{2p} - \begin{bmatrix} -\eta^* \\ \xi^* \end{bmatrix} \begin{bmatrix} -\eta & \xi \end{bmatrix} \begin{bmatrix} -\sigma_1 & 0 \\ 0 & \sigma_1 \end{bmatrix} \frac{1}{\mathbb{X}(\lambda + w^*)} = \\ &= \begin{bmatrix} I_p + \dfrac{\eta^*\eta\sigma_1}{\mathbb{X}(\lambda + w^*)} & \dfrac{\eta^*\xi\sigma_1}{\mathbb{X}(\lambda + w^*)} \\ -\dfrac{\xi^*\eta\sigma_1}{\mathbb{X}(\lambda + w^*)} & I_p - \dfrac{\xi^*\xi\sigma_1}{\mathbb{X}(\lambda + w^*)} \end{bmatrix} = \begin{bmatrix} \Theta_{11} & \Theta_{12} \\ \Theta_{21} & \Theta_{22} \end{bmatrix},\end{aligned}$$

where using the Lyapunov equation (3.10)

$$\begin{aligned} A\mathbb{X} + \mathbb{X}A^* &= \mathbb{X}(-w^* - w) = \\ &= -BJB^* = -\begin{bmatrix} -\eta & \xi \end{bmatrix} \begin{bmatrix} -\sigma_1 & 0 \\ 0 & \sigma_1 \end{bmatrix} \begin{bmatrix} -\eta^* \\ \xi^* \end{bmatrix} \end{aligned}$$

we find that

$$\mathbb{X} = \frac{\xi\sigma_1\xi^* - \eta\sigma_1\eta^*}{w + w^*}. \tag{3.13}$$

To the best of our knowledge, Theorem 3.5 below is new. It will be of much use in the following sections.

Theorem 3.5. *Let $S_0 \in \boldsymbol{\mathcal{RS}}$ with minimal realization*

$$S_0(\lambda) = I_p - B_0^*\mathbb{X}_0^{-1}(\lambda I - A_0)^{-1}B_0\sigma_1, \tag{3.14}$$

and let Θ be given by (3.12). Then $S = T_\Theta(S_0) \in \boldsymbol{\mathcal{RS}}$ and a minimal realization of S is given by

$$S(\lambda) = I_p - B_S^*\mathbb{X}_S^{-1}(\lambda I - A_S)^{-1}B_S\sigma_1,$$

where

$$B_S = \begin{bmatrix} B_0 \\ \eta - \xi \end{bmatrix} \tag{3.15}$$

$$\mathbb{X}_S = \begin{bmatrix} \mathbb{X}_0 & 0 \\ 0 & \mathbb{X} \end{bmatrix} = \begin{bmatrix} \mathbb{X}_0 & 0 \\ 0 & \dfrac{\xi\sigma_1\xi^* - \eta\sigma_1\eta^*}{w + w^*} \end{bmatrix} \tag{3.16}$$

$$A_S = \begin{bmatrix} A_0 & \dfrac{B_0\sigma_1\xi^*}{\mathbb{X}} \\ -\eta\sigma_1 B_0^*\mathbb{X}_0^{-1} & -w^* - \dfrac{\eta\sigma_1(\eta^* - \xi^*)}{\mathbb{X}} \end{bmatrix} \tag{3.17}$$

Proof: From the definition

$$\begin{aligned} S(\lambda) &= T_\Theta(S_0(\lambda)) = \\ &= (\Theta_{11} + S_0(\lambda)\Theta_{21})^{-1}(\Theta_{12} + S_0(\lambda)\Theta_{22}) = \\ &= (I_p + \frac{\eta^*\eta\sigma_1}{\mathbb{X}(\lambda + w^*)} - S_0(\lambda)\frac{\xi^*\eta\sigma_1}{\mathbb{X}(\lambda + w^*)})^{-1} \times \\ &\quad \times \left(\frac{\eta^*\xi\sigma_1}{\mathbb{X}(\lambda + w^*)} + S_0(\lambda)[I_p - \frac{\xi^*\xi\sigma_1}{\mathbb{X}(\lambda + w^*)}] \right). \end{aligned}$$

Let us denote here $S_0(\lambda)\xi^* = \alpha^*$; then the preceding expression becomes

$$\begin{aligned} S(\lambda) &= (I_p + \frac{\eta^*\eta\sigma_1}{\mathbb{X}(\lambda+w^*)} - \frac{\alpha^*\eta\sigma_1}{\mathbb{X}(\lambda+w^*)})^{-1}\times \\ &\quad \times(\frac{\eta^*\xi\sigma_1}{\mathbb{X}(\lambda+w^*)} + S_0(\lambda) - \frac{\alpha^*\xi\sigma_1}{\mathbb{X}(\lambda+w^*)}) = \\ &= (I_p + \frac{(\eta^*-\alpha^*)\eta\sigma_1}{\mathbb{X}(\lambda+w^*)})^{-1}(S_0(\lambda) + \frac{(\eta^*-\alpha^*)\xi\sigma_1}{\mathbb{X}(\lambda+w^*)}) = \end{aligned}$$

Using $(I+b^*a)^{-1} = I - b^*a\dfrac{1}{1+ab^*}$ we get

$$\begin{aligned} S(\lambda) &= (I_p - \frac{(\eta^*-\alpha^*)\eta\sigma_1}{\mathbb{X}(\lambda+w^*)}\frac{1}{1+\dfrac{\eta\sigma_1(\eta^*-\alpha^*)}{\mathbb{X}(\lambda+w^*)}})(S_0(\lambda) + \frac{(\eta^*-\alpha^*)\xi\sigma_1}{\mathbb{X}(\lambda+w^*)}) = \\ &= S_0(\lambda) - \frac{(\eta^*-S_0(\lambda)\xi^*)[\eta\sigma_1 S_0(\lambda) - \xi\sigma_1]}{\mathbb{X}(\lambda+w^*) + \eta\sigma_1(\eta^* - S_0(\lambda)\xi^*)}. \end{aligned}$$

Suppose further that there is a realization of the form (3.14) for the function $S_0(\lambda)$. Inserting it here we shall obtain

$$\begin{aligned} S(\lambda) &= S_0(\lambda) - \frac{(\eta^*-S_0(\lambda)\xi^*)[\eta\sigma_1 S_0(\lambda) - \xi\sigma_1]}{\mathbb{X}(\lambda+w^*) + \eta\sigma_1(\eta^* - S_0(\lambda)\xi^*)} = \\ &= I_p - B_0^*\mathbb{X}_0^{-1}(\lambda I - A_0)^{-1}B_0\sigma_1 - \\ &\quad - \frac{(\eta^*-S_0(\lambda)\xi^*)[\eta\sigma_1 S_0(\lambda) - \xi\sigma_1]}{\mathbb{X}(\lambda+w^*) + \eta\sigma_1(\eta^* - S_0(\lambda)\xi^*)}. \end{aligned}$$

Let us denote

$$\begin{aligned} M &= \mathbb{X}(\lambda+w^*) + \eta\sigma_1(\eta^* - S_0(\lambda)\xi^*) = \\ &= \mathbb{X}(\lambda+w^*) + \eta\sigma_1(\eta^*-\xi^*) + B_0^*\mathbb{X}_0^{-1}(\lambda I - A_0)^{-1}B_0\sigma_1\xi^*. \end{aligned}$$

The preceding formula for $S(\lambda)$ becomes

$$\begin{aligned} S(\lambda) &= I_p - B_0^*\mathbb{X}_0^{-1}(\lambda I - A_0)^{-1}B_0\sigma_1 - \frac{(\eta^*-S_0(\lambda)\xi^*)[\eta\sigma_1 S_0(\lambda) - \xi\sigma_1]}{\mathbb{X}(\lambda+w^*) + \eta\sigma_1(\eta^* - S_0(\lambda)\xi^*)} = \\ &= I_p - \frac{1}{M}\begin{bmatrix} B_0^* & \eta^*-\xi^* \end{bmatrix}\begin{bmatrix} \mathbb{X}_0^{-1} & 0 \\ 0 & 1 \end{bmatrix}\times \\ &\quad \times\begin{bmatrix} \alpha^{-1}M + \alpha^{-1}\beta\delta\alpha^{-1} & -\alpha^{-1}\beta \\ -\delta\alpha^{-1} & 1 \end{bmatrix}\begin{bmatrix} B_0 \\ \eta-\xi \end{bmatrix}\sigma_1, \end{aligned}$$

where

$$\begin{aligned} \alpha &= \lambda I - A_0 \\ \beta &= -B_0\sigma_1\xi^* \\ \delta &= \eta\sigma_1 B_0^*\mathbb{X}_0^{-1}. \end{aligned}$$

By definition of M we have

$$M = \mathbb{X}(\lambda+w^*) + \eta\sigma_1(\eta^*-\xi^*) - \delta\alpha^{-1}\beta,$$

and hence we obtain from the formula

$$\begin{bmatrix} \alpha & \beta \\ \delta & D \end{bmatrix}^{-1} = \frac{1}{D-\delta\alpha^{-1}\beta} \begin{bmatrix} \alpha^{-1}(D-\delta\alpha^{-1}\beta)+\alpha^{-1}\beta\delta\alpha^{-1} & -\alpha^{-1}\beta \\ -\delta\alpha^{-1} & 1 \end{bmatrix}$$

that the last expression for $S(\lambda)$ is

$$\begin{aligned} S(\lambda) &= I_p - \frac{1}{M}\begin{bmatrix} B_0^* & \eta^*-\xi^* \end{bmatrix} \begin{bmatrix} \mathbb{X}_0^{-1} & 0 \\ 0 & 1 \end{bmatrix} \times \\ &\quad \times \begin{bmatrix} \alpha^{-1}M+\alpha^{-1}\beta\delta\alpha^{-1} & -\alpha^{-1}\beta \\ -\delta\alpha^{-1} & 1 \end{bmatrix} \begin{bmatrix} B_0 \\ \eta-\xi \end{bmatrix} \sigma_1 = \\ &= I_p - \begin{bmatrix} B_0^* & \eta^*-\xi^* \end{bmatrix} \begin{bmatrix} \mathbb{X}_0^{-1} & 0 \\ 0 & 1 \end{bmatrix} \times \\ &\quad \times \begin{bmatrix} \alpha & \beta \\ \delta & \mathbb{X}(\lambda+w^*)+\eta\sigma_1(\eta^*-\xi^*) \end{bmatrix}^{-1} \begin{bmatrix} B_0 \\ \eta-\xi \end{bmatrix} \sigma_1 = \\ &= I_p - \begin{bmatrix} B_0^* & \eta^*-\xi^* \end{bmatrix} \begin{bmatrix} \mathbb{X}_0^{-1} & 0 \\ 0 & \mathbb{X}^{-1} \end{bmatrix} \times \\ &\quad \times \begin{bmatrix} \lambda I - A_0 & -\dfrac{B_0\sigma_1\xi^*}{\mathbb{X}} \\ \eta\sigma_1 B_0^*\mathbb{X}_0^{-1} & \lambda+w^*+\dfrac{\eta\sigma_1(\eta^*-\xi^*)}{\mathbb{X}} \end{bmatrix}^{-1} \begin{bmatrix} B_0 \\ \eta-\xi \end{bmatrix} \sigma_1. \end{aligned}$$

Thus we have obtained the realization (3.15)–(3.17). Furthermore, the Lyapunov equation (3.10) holds since

$$\begin{aligned} & A_S\mathbb{X}_S + \mathbb{X}_S A_S^* = \\ &= \begin{bmatrix} A_0 & \dfrac{B_0\sigma_1\xi^*}{\mathbb{X}} \\ -\eta\sigma_1 B_0^*\mathbb{X}_0^{-1} & -w^* - \dfrac{\eta\sigma_1(\eta^*-\xi^*)}{\mathbb{X}} \end{bmatrix} \begin{bmatrix} \mathbb{X}_0 & 0 \\ 0 & \mathbb{X} \end{bmatrix} + \\ &\quad + \begin{bmatrix} \mathbb{X}_0 & 0 \\ 0 & \mathbb{X} \end{bmatrix} \begin{bmatrix} A_0^* & -\mathbb{X}_0^{-1}B_0\sigma_1\eta^* \\ \dfrac{\xi\sigma_1 B_0^*}{\mathbb{X}} & -w - \dfrac{(\eta^*-\xi^*)\sigma_1\eta^*}{\mathbb{X}} \end{bmatrix} \\ &= \begin{bmatrix} A_0\mathbb{X}_0+\mathbb{X}_0A_0^* & -B_0\sigma_1(\eta^*-\xi^*) \\ -(\eta-\xi)\sigma_1 B_0^* & -\mathbb{X}(w+w^*)-\eta\sigma_1(\eta^*-\xi^*)-(\eta^*-\xi^*)\sigma_1\eta^* \end{bmatrix} \\ &= \begin{bmatrix} A_0\mathbb{X}_0+\mathbb{X}_0A_0^* & -B_0\sigma_1(\eta^*-\xi^*) \\ -(\eta-\xi)\sigma_1 B_0^* & \eta\sigma_1\eta^*-\xi\sigma_1\xi^*-\eta\sigma_1(\eta^*-\xi^*)-(\eta^*-\xi^*)\sigma_1\eta^* \end{bmatrix} \\ &= -B_S\sigma_1 B_S^*. \end{aligned}$$

The last equality follows easily from the Lyapunov equation for the given realization of S_0 and the formula (3.13). $\square$

4. The tangential Schur algorithm in the class $\boldsymbol{\mathcal{RSI}}$

Our strategy to the tangential Schur algorithm in the class $\boldsymbol{\mathcal{RSI}}$ relies on the following theorem. This theorem shows in particular that one cannot use the naive approach of applying the classical tangential Schur algorithm for each t_2 (that is, looking at t_2 as a mere parameter). The theorem itself is stated and proved for the non-rational case.

Theorem 4.1. *Let us fix the parameters σ_1, σ_2, and γ, and the interval* I*. Then for every $t_2^0 \in \mathrm{I}$ there is a one-to-one correspondence between pairs (γ_*, S) such that $S \in \boldsymbol{\mathcal{SI}}$ and γ_* continuous in a neighborhood of t_2^0, and functions $Y(\lambda) \in \boldsymbol{\mathcal{S}}(t_2^0)$.*

Proof: Let ϕ be the map which to a pair (γ_*, S) associates the function $S(\lambda, t_2^0) \in \boldsymbol{\mathcal{S}}(t_2^0)$. The converse map $\psi : Y(\lambda) \to Y(\lambda, t_2)$ was introduced in [M], [MV1, chapter 7] in a more general setting, and is defined as follows. Suppose that we have realized the transfer function $Y(\lambda)$ in the form

$$Y(\lambda) = I_p - B_0^* \mathbb{X}_0^{-1} (\lambda I - A_1)^{-1} B_0 \sigma_1(t_2^0).$$

Then we construct $B(t_2)$ from the differential equation with the spectral matrix parameter A_1:

$$\frac{d}{dt_2}[B(t_2)\sigma_1(t_2)] + A_1 B(t_2)\sigma_2(t_2) + B(t_2)\gamma(t_2) = 0, \quad B(t_2^0) = B_0. \tag{4.1}$$

In fact, the function $B(t_2)$ is given by the formula

$$B(t_2) = \oint (\lambda I - A_1)^{-1} B_0 \sigma_1^{-1} \Phi^{-1}(\lambda, t_2, t_2^0) d\lambda. \tag{4.2}$$

Next we construct $\mathbb{X}(t_2)$ on the maximal interval $\mathcal{I}$, where it is invertible (2.14) via the formula

$$\frac{d}{dt_2}\mathbb{X}(t_2) = B(t_2)\sigma_2(t_2)B(t_2)^*, \quad \mathbb{X}(t_2^0) = \mathbb{X}_0. \tag{4.3}$$

Finally, we define

$$\begin{aligned} \gamma_*(t_2) \ &= \gamma(t_2) + \sigma_2(t_2)B(t_2)^*\mathbb{X}^{-1}(t_2)B(t_2)\sigma_1(t_2) - \\ &\quad - \sigma_1(t_2)B(t_2)^*\mathbb{X}^{-1}(t_2)B(t_2)\sigma_2(t_2). \end{aligned}$$

Then easy computations show that the function

$$S(\lambda, t_2) = I_p - B(t_2)^*\mathbb{X}^{-1}(t_2)(\lambda I - A_1)^{-1}B(t_2)\sigma_1(t_2) \tag{4.4}$$

is in the class $\boldsymbol{\mathcal{CI}}$ corresponding to the parameters $\sigma_1(t_2)$,$\sigma_2(t_2)$,$\gamma(t_2)$, and $\gamma_*(t_2)$ for $t_2 \in \mathrm{I}$. Moreover, the Lyapunov equation (3.8) holds for

every $t_2 \in \mathrm{I}$, and thus the realization (4.4) is minimal for every $t_2 \in \mathrm{I}$.

Note that the composition $\phi \circ \psi = id$, since starting from a function $Y \in \boldsymbol{\mathcal{S}}(t_2^0)$, constructing $Y(\lambda, t_2)$ and taking its value at t_2^0, we shall obtain again Y from the initial conditions of the differential equations (4.1) and (4.3) defining $B(t_2), \mathbb{X}(t_2)$.

In order to show that $\psi \circ \phi = id$, we start from a function $S(\lambda, t_2) \in \boldsymbol{\mathcal{SI}}$ and take its value $S(\lambda, t_2^0) \in \boldsymbol{\mathcal{S}}(t_2^0)$. Using the construction above, we shall obtain a function $Y(\lambda, t_2)$. Note that the two functions $S(\lambda, t_2)$ and $Y(\lambda, t_2)$ have the same value at t_2^0 and maps solutions of the same input LDE to (possibly different) output LDEs, i.e. :

$$\begin{aligned} S(\lambda, t_2) &= \Phi_*(\lambda, t_2, t_2^0) S(\lambda, t_2^0) \Phi^{-1}(\lambda, t_2, t_2^0), \\ Y(\lambda, t_2) &= \Phi'_*(\lambda, t_2, t_2^0) S(\lambda, t_2^0) \Phi^{-1}(\lambda, t_2, t_2^0). \end{aligned}$$

Then the function $S^{-1}(\lambda, t_2) Y(\lambda, t_2)$ is equal to I_p at infinity and is entire. By Liouville's theorem it is a constant function and is equal to I_p. Thus

$$\Phi_*(\lambda, t_2, t_2^0) = \Phi'_*(\lambda, t_2, t_2^0),$$

from where we obtain that

$$\Phi_*^{-1}(\lambda, t_2, t_2^0) \Phi'_*(\lambda, t_2, t_2^0) = I_p.$$

Differentiating both sides of this last equation we get to

$$\begin{aligned} 0 &= \frac{\partial}{\partial t_2} [\Phi_*^{-1}(\lambda, t_2, t_2^0) \Phi'_*(\lambda, t_2, t_2^0)] = \\ &= \Phi_*^{-1}(\lambda, t_2, t_2^0) \sigma_1^{-1}(t_2)(-\gamma(t_2) + \gamma'(t_2)) \Phi'_*(\lambda, t_2, t_2^0). \end{aligned}$$

Since the matrices $\Phi_*(\lambda, t_2, t_2^0), \Phi'_*(\lambda, t_2, t_2^0), \sigma_1(t_2)$ are invertible we obtain that $\gamma(t_2) = \gamma'(t_2)$. $\square$

Remark: Last theorems claims that the correspondence is between "initial" values $S(\lambda, t_2^0)$ and pairs $(S(\lambda, t_2), \gamma_*(t_2))$. Notice that it is possible to obtain functions with the same $\gamma_*(t_2)$ with different initial conditions:

Proposition 4.2. *Suppose that there exists a function $Y \in \boldsymbol{\mathcal{S}}(t_2^0)$, which commutes with $\Phi(\lambda, t_2, t_2^0)$ and suppose that a function $S \in \boldsymbol{\mathcal{SI}}$ corresponds to certain vessel parameters $\sigma_1, \sigma_2, \gamma, \gamma_*$. Then the function SY belongs to the class $\boldsymbol{\mathcal{SI}}$ and corresponds to the same vessel parameters $\sigma_1, \sigma_2, \gamma, \gamma_*$.*

Proof: Using formula 2.21 we obtain that

$$S(\lambda, t_2) = \Phi_*(\lambda, t_2, t_2^0) S(\lambda, t_2^0) \Phi^{-1}(\lambda, t_2, t_2^0).$$

Consequently,

$$S(\lambda,t_2)Y(\lambda)=\Phi_*(\lambda,t_2,t_2^0)S(\lambda,t_2^0)Y(\lambda)\Phi^{-1}(\lambda,t_2,t_2^0)$$

intertwines solutions of the input (2.16) and the output (2.17) ODEs with the spectral parameter λ, and is identity at infinity, because S and Y and their product are such. Thus by the definition the function $SY\in\boldsymbol{\mathcal{SI}}$ and corresponds to the same spectral parameters as S. $\square$

For example (to be studied in subsection 5.2), taking

$$\sigma_1=\begin{bmatrix}0&1\\1&0\end{bmatrix},\sigma_2=\begin{bmatrix}1&0\\0&0\end{bmatrix},\gamma=\begin{bmatrix}0&0\\0&i\end{bmatrix}$$

we shall obtain that

$$\Phi(\lambda,t_2,t_2^0)=V^{-1}\begin{bmatrix}e^{-k(t_2-t_2^0)}&0\\0&e^{k(t_2-t_2^0)}\end{bmatrix}V,$$

where

$$V=\begin{bmatrix}-\dfrac{\sqrt{2}k}{\lambda}&1\\\dfrac{\sqrt{2}k}{\lambda}&1\end{bmatrix},\quad k=\frac{(1+i)\sqrt{\lambda}}{2}.$$

Taking Y, which commutes with $\sigma_1^{-1}(\sigma_2\lambda+\gamma)=\begin{bmatrix}0&i\\\lambda&0\end{bmatrix}$, i.e. of the form

$$Y(\lambda)=I_2-\begin{bmatrix}a(\lambda)&\dfrac{ic(\lambda)}{\lambda}\\c(\lambda)&a(\lambda)\end{bmatrix}$$

we shall obtain that for any $S\in\boldsymbol{\mathcal{SI}}$, the function $SY\in\boldsymbol{\mathcal{SI}}$ and corresponds to the same vessel parameters.

A stronger condition on $\gamma_*(t_2)$ will be presented in section 5, after we shall study Markov moments of the transfer functions.

We now focus on the rational case. We first note that starting from different realizations at t_2^0 we shall obtain the same $\gamma_*(t_2)$. More precisely we have the following theorem:

Theorem 4.3. *Suppose that there are two minimal realizations of the function $S(\lambda,t_2^0)\in\boldsymbol{\mathcal{RS}}$:*

$$S(\lambda,t_2^0)=I_p-B_\ell^*\mathbb{X}_\ell^{-1}(\lambda I-A_1)^{-1}B_\ell\sigma_1(t_2^0),\quad \ell=1,2,$$

with associated similarity matrix V. Then the functions

$$S_\ell(\lambda,t_2)=I_p-B_\ell(t_2)^*\mathbb{X}_\ell^{-1}(t_2)(\lambda I-A_\ell)^{-1}B_\ell(t_2)\sigma_1(t_2),\quad \ell=1,2,$$

obtained from these realizations via the construction in Theorem 4.1 are the same, and this holds if and only if:

$$B_2(t_2)=VB_1(t_2), \tag{4.5}$$

where t_2 varies in a neighborhood of t_2^0. Moreover, in this case the following formula holds

$$\mathbb{X}_2(t_2) = V\mathbb{X}_1(t_2)V^*.$$

Proof: Equality of the two realizations means that there exists an invertible matrix V such that

$$A_2 = VA_1V^{-1},\ B_2\sigma_1(t_2^0) = VB_1\sigma_1(t_2^0),\ B_1^*\mathbb{X}_1^{-1}V = B_2^*\mathbb{X}_2^{-1},$$

from which follows (see [AG, Lemma 2.1 p. 184] for instance) that

$$\mathbb{X}_2 = V\mathbb{X}_1V^*.$$

By the construction described in Theorem 4.1, the function $B_1(t_2)$ will satisfy (4.1)

$$\frac{d}{dt_2}[B_1(t_2)\sigma_1(t_2)] + A_1B_1(t_2)\sigma_2(t_2) + B_1(t_2)\gamma(t_2) = 0, \quad B_1(t_2^0) = B_1,$$

and the function $B_2(t_2)$ will satisfy the same equation with A_2 instead of A_1:

$$\frac{d}{dt_2}[B_2(t_2)\sigma_1(t_2)] + A_2B_2(t_2)\sigma_2(t_2) + B_2(t_2)\gamma(t_2) = 0, \quad B_2(t_2^0) = B_2.$$

Using the equalities $A_2 = VA_1V^{-1}$, $B_2 = VB_1$ we obtain that the function $V^{-1}B_2(t_2)$ satisfies

$$\begin{array}{l}\frac{d}{dt_2}[V^{-1}B_2(t_2)\sigma_1(t_2)] + A_1V^{-1}B_2(t_2)\sigma_2(t_2) + V^{-1}B_2(t_2)\gamma(t_2) = 0,\\ V^{-1}B_2(t_2^0) = B_1,\end{array}$$

which is the same differential equation as for $B_1(t_2)$. Thus $B_2(t_2) = VB_1(t_2)$. Similarly, considering the differential equations

$$\frac{d}{dt_2}\mathbb{X}_i(t_2) = B_i(t_2)\sigma_1B_i(t_2)^*, \quad \mathbb{X}_i(t_2^0) = \mathbb{X}_i, \quad i = 1, 2,$$

we obtain that

$$\mathbb{X}_2(t_2) = V\mathbb{X}_1(t_2)V^*.$$

Consequently,

$$\begin{array}{ll}B_2^*(t_2)\mathbb{X}_2^{-1}(t_2)B(t_2) & = B_1^*(t_2)V^*V^{-1*}\mathbb{X}_1^{-1}(t_2)V^{-1}VB_1(t_2)\\ & = B_1^*(t_2)\mathbb{X}_1^{-1}(t_2)B_1(t_2),\end{array}$$

from where we conclude that the same function γ_* is associated to $S_1(\lambda, t_2)$ and $S_2(\lambda, t_2)$. Since these functions coincide for $t_2 = t_2^0$ and map the same input ODE, we obtain that they are equal in a neighborhood of t_2^0. $\square$

The following notion has been introduced and studied in [M2].

Definition 4.4. *Let $S \in \boldsymbol{\mathcal{RSI}}$ with a realization* (4.4)*. The function*

$$\tau(t_2) = \det \mathbb{X}(t_2).$$

is called the τ-function associated to S.

It follows from Theorem 4.3 that the τ function is well defined up to a multiplicative strictly positive constant. Indeed using the notation of the theorem,

$$\det \mathbb{X}_2(t_2) = \det[V\mathbb{X}_1(t_2)V^*] = \det \mathbb{X}_1(t_2)\det(VV^*).$$

We now introduce the counterpart of the tangential Schur algorithm in the class $\boldsymbol{\mathcal{RSI}}$. Let $S \in \boldsymbol{\mathcal{RSI}}$ and fix $t_2^0 \in \mathrm{I}$. The function $S(\lambda, t_2^0) \in \boldsymbol{\mathcal{RS}}$. Consider now a space $\mathcal{M}$ with $\mathbb{X} > 0$ and corresponding function Θ as in Corollary 3.3. This is always possible in view of Lemma 3.4. It follows from Theorem 3.2 that there exists $S_0 \in \boldsymbol{\mathcal{RS}}$ such that

$$S(\lambda, t_2^0) = T_{\Theta(\lambda)}(S_0(\lambda)). \tag{4.6}$$

Applying Theorem 4.1 to S_0, we obtain a uniquely defined function $S_0(\lambda, t_2) \in \boldsymbol{\mathcal{RSI}}$, such that at t_2^0 the relation (4.6) holds.

Definition 4.5. *The map T_{Θ,t_2^0}*

$$S(\lambda, t_2) \mapsto S_0(\lambda, t_2)$$

is the time-varying counterpart of the linear fractional transformation (4.6)*. We will call it a generalized linear fractional transformation.*

If $S(\lambda, t_2) \in \boldsymbol{\mathcal{SI}}$ corresponds to the vessel parameters σ_1, σ_2, γ, γ_*, then $S_0(\lambda, t_2^0) \in \boldsymbol{\mathcal{SI}}$ corresponds to the vessel parameters σ_1, σ_2, γ, $\gamma_{0,*}$ for a uniquely defined function $\gamma_{0,*}(t_2)$. Moreover, for $t_2 = t_2^0$ we have the usual linear fractional transformation (4.6).

As a consequence of Theorem 4.1 the following lemma holds.

Lemma 4.6. *For a given J-inner function Θ and a given point t_2^0, the map T_{Θ,t_2^0} is one-to-one from $\boldsymbol{\mathcal{RS}}$ into $\boldsymbol{\mathcal{RSI}}$.*

Proof: Notice that for a given t_2^0 the map T_Θ is injective. Furthermore, using Theorem 4.1, every $S \in \boldsymbol{\mathcal{RSI}}$ is uniquely defined by the function $S(\lambda, t_2^0) \in \boldsymbol{\mathcal{RS}}$. The result follows. $\square$

Suppose now that we start from $S_0(\lambda) \equiv I_p$ and apply n linear fractional transformations for a fixed t_2^0, using the data $< w_i, \xi_i, \eta_i >$ $(i = 1, \dots n)$ to construct the corresponding J-inner functions. We obtain a function

$$S_n(\lambda) = T_\Theta(I) = I - B_n^* X_n^{-1}(\lambda I - A_n)^{-1} B_n \sigma_1 \in \boldsymbol{\mathcal{RS}}.$$

Using iteratively formulas (3.15), (3.16), (3.17) we obtain that

$$B_n = \begin{bmatrix} \eta_1 - \xi_1 \\ \vdots \\ \eta_n - \xi_n \end{bmatrix},$$

$$\mathbb{X}_n = \mathrm{diag}[\widetilde{\mathbb{X}}_1, \ldots, \widetilde{\mathbb{X}}_n],$$

and

(4.7)

$$A_n = \begin{bmatrix} -w_1^* - \frac{\eta_1\sigma_1(\eta_1^* - \xi_1^*)}{\widetilde{\mathbb{X}}_1} & \frac{(\eta_1 - \xi_1)\sigma_1\xi_2^*}{\widetilde{\mathbb{X}}_2} & \cdots & \frac{(\eta_1 - \xi_1)\sigma_1\xi_n^*}{\widetilde{\mathbb{X}}_n} \\ -\frac{\eta_2\sigma_1(\eta_1^* - \xi_1^*)}{\widetilde{\mathbb{X}}_1} & -w_2^* - \frac{\eta_2\sigma_1(\eta_2^* - \xi_2^*)}{\widetilde{\mathbb{X}}_2} & \cdots & \frac{(\eta_2 - \xi_2)\sigma_1\xi_n^*}{\widetilde{\mathbb{X}}_n} \\ \vdots & \vdots & \ddots & \vdots \\ -\frac{\eta_n\sigma_1(\eta_1^* - \xi_1^*)}{\widetilde{\mathbb{X}}_1} & -\frac{\eta_n\sigma_1(\eta_2^* - \xi_2^*)}{\widetilde{\mathbb{X}}_2} & \cdots & -w_n^* - \frac{\eta_n\sigma_1(\eta_n^* - \xi_n^*)}{\widetilde{\mathbb{X}}_n} \end{bmatrix},$$

where $\widetilde{\mathbb{X}}_i$ is defined by (3.13):

$$\widetilde{\mathbb{X}}_i = \frac{\xi_i\sigma_1\xi_i^* - \eta_i\sigma_1\eta_i^*}{w_i + w_i^*}, \quad i = 1, \ldots n.$$

Assume now that, starting from the identity matrix we apply this procedure for two different sets of data (with the same n) at two different points t_2^1 and t_2^2. The following theorem answers the question as when we obtain the same function, that is, when do we have:

$$T_{\Theta_1, t_2^1}(I_p) = T_{\Theta_2, t_2^2}(I_p).$$

Theorem 4.7. *Suppose that there are given two sets of n triples $< w_i^1, \xi_i^1, \eta_i^1 >$ and $< w_i^2, \xi_i^2, \eta_i^2 >$, with corresponding $\Theta_\ell, \ell = 1, 2$. Then necessary and sufficient conditions for equality of the two functions*

$$S_\ell(\lambda, t_2) = T_{\Theta_\ell, t_2^\ell}(I) = I_p - B_n^\ell(\mathbb{X}_n^\ell)^{-1}(\lambda I - A_n^\ell)^{-1}B_n^\ell\sigma_1, \quad \ell = 1, 2$$

are:

1. *The corresponding matrices A_n^1 and A_n^2 defined by (4.7) are similar, i.e. there exists an invertible matrix V such that $A_n^1 = VA_n^2V^{-1}$,*
2. $\oint(\lambda I - A_n^1)^{-1}B_n^1\sigma_1^{-1}\Phi^{-1}(\lambda, t_2^2, t_2^1)d\lambda = VB_n^2$.

Proof: From Theorem 4.1, a necessary and sufficient condition for the functions to be equal is that

$$S_1(\lambda, t_2^2) = S_2(\lambda, t_2^2).$$

From 4.3 this holds if and only if

$$A_n^2 = VA_n^1V^{-1}, \; B_n^2(t_2^2) = VB_n^1, \; \mathbb{X}_n^2(t_2^2) = V\mathbb{X}_n^1V^*$$

for a uniquely defined invertible matrix V. The result follows using formula (4.2). $\square$

A more general construction in this setting is obtained if one supposes that at each step different values of t_2 are chosen. In this case the construction of the function $S_n(\lambda, t_2)$ is more complicated, and can be computed recursively. The formulas are very involved in this case and we can see no real advantage to develop them at this point.

4.1. **Linear Fractional Transformations in terms of intertwining positive pairs.** Suppose that we are given a data of the NP interpolation problem 1.6. Following the notations of corollary 3.3 let us write

$$B(t_2) = \begin{bmatrix} -\xi_1(t_2)^* S(w_1, t_2)^* & \xi_1(t_2)^* \\ \vdots & \vdots \\ -\xi_n(t_2)^* S(w_n, t_2)^* & \xi_n(t_2)^* \end{bmatrix}, \ A_1 = \operatorname{diag}[-w_1^*, \dots, -w_n^*].$$

Let us denote by capital Greek letters the following vessel parameters

$$\Sigma_1(t_2) = \begin{bmatrix} -\sigma_1(t_2) & 0 \\ 0 & \sigma_1(t_2) \end{bmatrix} = J,$$
$$\Sigma_2(t_2) = \begin{bmatrix} \sigma_2(t_2) & 0 \\ 0 & \sigma_2(t_2) \end{bmatrix},$$
$$\Gamma(t_2) = \begin{bmatrix} \gamma_*(t_2) & 0 \\ 0 & \gamma(t_2) \end{bmatrix},$$

then simple calculations show that $B(t_2)$ satisfies (4.1)

$$\frac{d}{dt_2}[B(t_2)\Sigma_1(t_2)] + A_1 B(t_2)\Sigma_2(t_2) + B(t_2)\Gamma(t_2) = 0, \quad B(t_2^0) = B.$$

Suppose that $\mathbb{X}(t_2) > 0$ is a solution of

$$A_1\mathbb{X}(t_2) + \mathbb{X}(t_2)A_1^* + B(t_2)\Sigma_1(t_2)B^*(t_2) = 0,$$
$$\frac{d}{dt_2}\mathbb{X}(t_2) = B(t_2)\Sigma_2(t_2)B^*(t_2),$$

which is always possible if $\Re w_i \neq 0$ for each $i = 1, \dots, n$. Then the following collection

$$\mathfrak{V} = \{A_1, B(t_2), \mathbb{X}(t_2); \Sigma_1(t_2), \Sigma_2(t_2), \Gamma(t_2), \Gamma_*(t_2); \mathbb{C}^{2n}; \mathcal{E} \oplus \mathcal{E}\}.$$

is a vessel for $\Gamma_*(t_2)$ defined from the linkage condition (2.15)

$$\Gamma_*(t_2) = \Gamma(t_2) + \Sigma_1(t_2)B(t_2)^*\mathbb{X}^{-1}(t_2)B(t_2)\Sigma_2(t_2) - \\ -\Sigma_2(t_2)B(t_2)^*\mathbb{X}^{-1}(t_2)B(t_2)\Sigma_1(t_2).$$

Transfer function of the vessel $\mathfrak{V}$ is

$$\Theta(\lambda,t_2)=I_{2p}-B(t_2)^*\mathbb{X}^{-1}(t_2)(\lambda I_n-A_1)^{-1}B(t_2)\Sigma_1(t_2),$$

which is in $\boldsymbol{\mathcal{RSI}}(\mathbf{U}_*\oplus\mathbf{U},\widetilde{\mathbf{U}})$ for

$$\widetilde{\mathbf{U}}=\lambda\Sigma_2(t_2)-\Sigma_1(t_2)\frac{d}{dt_2}+\Gamma_*(t_2).$$

If we denote further the decomposition of $\Theta(t_2)$ as

$$\Theta(\lambda,t_2)=\begin{bmatrix}\Theta_{11}(\lambda,t_2) & \Theta_{12}(\lambda,t_2)\\ \Theta_{21}(\lambda,t_2) & \Theta_{22}(\lambda,t_2)\end{bmatrix}$$

then if one defines $S_0(\lambda,t_2)$ so that

$$(\Theta_{11}+S_0\Theta_{21})^{-1}(\Theta_{12}+S_0\Theta_{22})=S,$$

then the function $S(\lambda,t_2)$ usually does not intertwine solutions of LDEs with spectral parameter λ.
Instead, we define

$$W(\lambda,t_2)=\begin{bmatrix}W_1(\lambda,t_2) & W_2(\lambda,t_2)\end{bmatrix}$$

so that

$$W_1\Theta_{11}+W_2\Theta_{21}=I_p,\quad W_1\Theta_{12}+W_2\Theta_{22}=S,$$

then the following lemma holds

Lemma 4.8. *The pair of functions $W(\lambda,t_2)$ is in $\boldsymbol{\mathcal{SI}}(\widetilde{\mathbf{U}},\mathbf{U}_*)$ and $\dfrac{W(\lambda,t_2)^*\Sigma_1(t_2)W(\mu,t_2)}{\bar{\lambda}+\mu}\geq 0$ on the domain of analyticity of $W(\lambda,t_2)$.*

Proof: Since $\Theta(\lambda,t_2)$ is invertible for all λ out of the spectrum of A_1, an element of $\widetilde{U}$ is of the form $\Theta(\lambda,t_2)\begin{bmatrix}y_\lambda(t_2)\\ u_\lambda(t_2)\end{bmatrix}$, where $y_\lambda(t_2)$, $u_\lambda(t_2)$ satisfy (2.17) and (2.16) respectively. Then

$$\begin{aligned}W(\lambda,t_2)\Theta(\lambda,t_2)\begin{bmatrix}y_\lambda(t_2)\\ u_\lambda(t_2)\end{bmatrix}&=\\ =\begin{bmatrix}W_1(\lambda,t_2) & W_2(\lambda,t_2)\end{bmatrix}\begin{bmatrix}\Theta_{11}(\lambda,t_2) & \Theta_{12}(\lambda,t_2)\\ \Theta_{21}(\lambda,t_2) & \Theta_{22}(\lambda,t_2)\end{bmatrix}\begin{bmatrix}y_\lambda(t_2)\\ u_\lambda(t_2)\end{bmatrix}&=\\ =\begin{bmatrix}W_1\Theta_{11}+W_2\Theta_{21} & W_1\Theta_{12}+W_2\Theta_{22}\end{bmatrix}\begin{bmatrix}y_\lambda(t_2)\\ u_\lambda(t_2)\end{bmatrix}&=\\ =\begin{bmatrix}I & S(\lambda,t_2)\end{bmatrix}\begin{bmatrix}y_\lambda(t_2)\\ u_\lambda(t_2)\end{bmatrix}=y_\lambda(t_2)+S(\lambda,t_2)u_\lambda(t_2)\in\mathbf{U}_*&,\end{aligned}$$

since $y_\lambda(t_2)$ and $S(\lambda,t_2)u_\lambda(t_2)$ are in $\mathbf{U}_*$.

From the formula $W\Theta = \begin{bmatrix} I & S \end{bmatrix}$, it follows that

$$W(\lambda, t_2) = \begin{bmatrix} I & S(\lambda, t_2) \end{bmatrix} \Theta^{-1}(\lambda, t_2).$$

Consequently, the expression $\dfrac{W(\lambda, t_2)^* \Sigma_1(t_2) W(\mu, t_2)}{\bar{\lambda} + \mu}$ considered on the domain of analyticity of $W(\cdot, t_2)$ becomes

$$\frac{W(\lambda, t_2)\Sigma_1(t_2)W^*(\mu, t_2)}{\lambda + \bar{\mu}} =$$
$$= \begin{bmatrix} I & S(\lambda, t_2) \end{bmatrix} \frac{\Theta^{-1}(\lambda, t_2)\Sigma_1(t_2)\Theta^{-1*}(\mu, t_2)}{\lambda + \bar{\mu}} \begin{bmatrix} I \\ S^*(\mu, t_2) \end{bmatrix}.$$

Since $\Theta(\lambda, t_2)$ is a transfer function of a conservative vessel, its inverse is a transfer function too and satisfies

$$\frac{\Theta^{-1}(\lambda, t_2)\Sigma_1(t_2)\Theta(\mu, t_2)^{-1*}}{\lambda + \bar{\mu}} \geq \Sigma_1(t_2)$$

and consequently,

$$\begin{aligned} \frac{W(\lambda, t_2)\Sigma_1(t_2)W^*(\mu, t_2)}{\lambda + \bar{\mu}} & \geq \begin{bmatrix} I & S(\lambda, t_2) \end{bmatrix} \Sigma_1(t_2) \begin{bmatrix} I \\ S^*(\mu, t_2) \end{bmatrix} \\ & \geq \begin{bmatrix} I & S(\lambda, t_2) \end{bmatrix} \begin{bmatrix} -\sigma_1(t_2) & 0 \\ 0 & \sigma_1(t_2) \end{bmatrix} \begin{bmatrix} I \\ S^*(\mu, t_2) \end{bmatrix} \\ & \geq S(\lambda, t_2)\sigma_1(t_2)S^*(\mu, t_2) - \sigma_1(t_2) \end{aligned}$$

Notice that

$$\begin{aligned} & W(\lambda, t_2)\Sigma_1(t_2)W(-\bar{\lambda}, t_2) = \\ & = \begin{bmatrix} I & S(\lambda, t_2) \end{bmatrix} \Theta^{-1}(\lambda, t_2)\Sigma_1(t_2)\Theta^{-1*}(-\bar{\lambda}, t_2) \begin{bmatrix} I \\ S^*(-\bar{\lambda}, t_2) \end{bmatrix} = \\ & = \begin{bmatrix} I & S(\lambda, t_2) \end{bmatrix} \Sigma_1(t_2) \begin{bmatrix} I \\ S^*(-\bar{\lambda}, t_2) \end{bmatrix} = \\ & = S(\lambda, t_2)\sigma_1(t_2)S^*(-\bar{\lambda}, t_2) - \sigma_1(t_2) = 0, \end{aligned}$$

by the properties of transfer functions for vessels. □

As a consequence of this theorem, we define

Definition 4.9. *A pair of functions*

$$W(\lambda, t_2) = \begin{bmatrix} W_1(\lambda, t_2) & W_2(\lambda, t_2) \end{bmatrix}$$

is called ***positive*** *if the conditions of lemma 4.8 hold:*

$$W(\lambda, t_2) \in \boldsymbol{\mathcal{SI}}(\widetilde{\mathbf{U}}, \mathbf{U}_*), \quad W(\lambda, t_2)JW(\lambda, t_2) \geq 0 \text{ on } \mathbb{C}_+$$

5. Markov moments and partial realization problem in the class $\boldsymbol{\mathcal{RSI}}$

Let $S \in \boldsymbol{\mathcal{RS}}$, and consider the Laurent expansion at infinity

$$S(\lambda) = I_p - B^*\mathbb{X}^{-1}(\lambda I - A)^{-1}B\sigma_1 = I_p - \sum_{i=0}^{\infty} \frac{1}{\lambda^{i+1}} B^*\mathbb{X}^{-1}A^iB\sigma_1$$

in terms of a given minimal realization. The matrices $H_i = B^*\mathbb{X}^{-1}A^iB\sigma_1$ are called the *Markov moments* of S. The Partial Realization Problem (or moment problem at infinity) in $\boldsymbol{\mathcal{RS}}$ is defined as follows: Given the first $n+1$ Markov moments $H_0, \dots, H_n$, find all functions (if any) $S \in \boldsymbol{\mathcal{RS}}$ with these first $n+1$ moments. See [GKL] for a general study of the partial realization problem. Similarly, one can define the Markov moments for an element $S \in \boldsymbol{\mathcal{RSI}}$. We now give necessary conditions which the moments of a function $S \in \boldsymbol{\mathcal{RSI}}$ have to satisfy. Then we consider the partial realization Problem 1.4, that is, given $n+1$ $\mathbb{C}^{p\times p}$-valued functions $H_0(t_2), \dots, H_n(t_2)$, for a fixed choice of σ_1, σ_2, find all $S \in \boldsymbol{\mathcal{RSI}}$ with these first $n+1$ moments.

It is important to notice the following: fixing $t_2 = t_2^0$ and solving the corresponding classical moment problem will not lead to a solution of the problem in the class $\boldsymbol{\mathcal{RSI}}$ because we cannot obtain the function γ (and hence γ_*) from this solution.

5.1. Restrictions on Markov moments for functions in $\boldsymbol{\mathcal{RSI}}$.

We study the Markov moments of a function $S \in \boldsymbol{\mathcal{RSI}}$, which maps solutions of the input ODE (2.16) to the output ODE (2.17) using a minimal realization (4.4) of S. First we develop an analogue of the formula .... for $2D$ vessels with constant coefficients.

Theorem 5.1. *For fixed $\sigma_1, \sigma_2, \gamma$, a necessary condition on γ_* so that there exists a vessel with the vessel parameters $\sigma_1, \sigma_2, \gamma, \gamma_*$ is*

$$\det \Phi_*(\lambda, t_2, t_2^0) = \det \Phi(\lambda, t_2, t_2^0)$$

Proof: Let $S \in \boldsymbol{\mathcal{SI}}$ be a function corresponding to the parameters $\sigma_1, \sigma_2, \gamma, \gamma_*$. Then using a realization (4.4)

$$S(\lambda, t_2) = I_p - B(t_2)^*\mathbb{X}^{-1}(t_2)(\lambda I - A_1)^{-1}B(t_2)\sigma_1(t_2)$$

and Lyapunov equation (3.8) we shall obtain that

$$\begin{aligned}\det S(\lambda,t_2) &= \det\big(I_p - B(t_2)^*\mathbb{X}^{-1}(t_2)(\lambda I - A_1)^{-1}B(t_2)\sigma_1(t_2)\big)\\ &= \det\big(I - B(t_2)\sigma_1(t_2)B(t_2)^*\mathbb{X}^{-1}(t_2)(\lambda I - A_1)^{-1}\big)\\ &= \det\big(I + (A_1\mathbb{X}(t_2)+\mathbb{X}(t_2)A_1^*)\mathbb{X}^{-1}(t_2)(\lambda I - A_1)^{-1}\big)\\ &= \det\big(I + A_1(\lambda I - A_1)^{-1} + \mathbb{X}(t_2)A_1^*\mathbb{X}^{-1}(t_2)(\lambda I - A_1)^{-1}\big)\\ &= \det\left((\lambda I + \mathbb{X}(t_2)A_1^*\mathbb{X}^{-1}(t_2))(\lambda I - A_1)^{-1}\right)\\ &= \det(\lambda I + A_1^*)\det(\lambda I - A_1)^{-1}\\ &= \det S(\lambda, t_2^0).\end{aligned}$$

Consequently, taking determinant of the formula (2.21)

$$S(\lambda,t_2)\Phi(\lambda,t_2,t_2^0) = \Phi_*(\lambda,t_2,t_2^0)S(\lambda,t_2^0)$$

we shall obtain that $\det\Phi_*(\lambda,t_2,t_2^0) = \det\Phi(\lambda,t_2,t_2^0)$ for all points λ, where $\det S(\lambda,t_2^0)$ exists and is different from zero. Since it happens for all points outside the spectrum of A_1 and the functions $\det\Phi_*(\lambda,t_2,t_2^0)$, $\det\Phi(\lambda,t_2,t_2^0)$ are entire they are equal for all λ. □

Since A_1 is a constant matrix, at some stage the elements $I, A_1, \dots, A_1^n$ will be linearly dependent and we obtain:

Lemma 5.2. *Given $S \in \boldsymbol{\mathcal{RSI}}$, with Markov moments $H_i(t_2)$, $i = 0,1,\dots$. Then exists N and constants μ_j, $j=1,\dots,N$ such that*

$$\sum_{j=0}^{N+1}\mu_j H_{n-j}(t_2) = 0, \quad n \ge N+1. \tag{5.1}$$

We notice that the first moment $H_0(t_2)$ satisfies the linkage condition (1.2):

$$\gamma_*(t_2) - \gamma(t_2) = \sigma_2(t_2)H_0(t_2) - \sigma_1(t_2)H_0(t_2)\sigma_1^{-1}(t_2)\sigma_2(t_2).$$

Let us denote

$$H_0(t_2) = C(t_2)B(t_2)\sigma_1(t_2) = (B(t_2))^*X^{-1}(t_2)B(t_2)\sigma_1(t_2).$$

The functions $B(t_2), C(t_2)$ satisfy the following differential equations

$$\frac{d}{dt_2}[B(t_2)\sigma_1(t_2)] + A_1B(t_2)\sigma_2(t_2) + B(t_2)\gamma(t_2) = 0 \tag{5.2}$$

$$\sigma_1(t_2)\frac{d}{dt_2}C(t_2) - \sigma_2(t_2)C(t_2)A_1 - \gamma_*(t_2)C(t_2) = 0, \tag{5.3}$$

see [M, MV1]. Thus, differentiating $H_0(t_2)$, we obtain

$$\begin{aligned}\tfrac{d}{dt_2}H_0(t_2) &= \tfrac{d}{dt_2}[C(t_2)B(t_2)\sigma_1(t_2)]\\ &= \sigma_1^{-1}(t_2)\sigma_2(t_2)C(t_2)A_1B(t_2)\sigma_1(t_2) - C(t_2)A_1B(t_2)\sigma_2(t_2) +\\ &\quad +\sigma_1^{-1}(t_2)\gamma_*(t_2)H_0(t_2) - H_0(t_2)\sigma_1^{-1}(t_2)\gamma(t_2).\end{aligned}$$

In other words the second moment $H_1(t_2) = C(t_2)A_1B(t_2)\sigma_1(t_2)$ satisfies the following differential equation

$$\begin{array}{l}\sigma_1^{-1}(t_2)\sigma_2(t_2)H_1(t_2) - H_1(t_2)\sigma_1^{-1}(t_2)\sigma_2(t_2) = \\ = \frac{d}{dt_2}H_0(t_2) - \sigma_1^{-1}\gamma_* H_0(t_2) + H_0(t_2)\sigma_1^{-1}(t_2)\gamma(t_2).\end{array}$$

In the same manner the moments $H_i(t_2)$ and $H_{i+1}(t_2)$ are connected by the following differential equation

$$\begin{array}{l}\sigma_1^{-1}(t_2)\sigma_2(t_2)H_{i+1} - H_{i+1}(t_2)\sigma_1^{-1}(t_2)\sigma_2(t_2) = \\ = \frac{d}{dt_2}H_i(t_2) - \sigma_1^{-1}(t_2)\gamma_*(t_2)H_i(t_2) + H_i(t_2)\sigma_1^{-1}(t_2)\gamma(t_2).\end{array} \tag{5.4}$$

Notice also that

$$\begin{array}{rl} H_i\sigma_1^{-1}H_0^*\sigma_1 & = B^*X^{-1}A_1^iB\sigma_1B^*X^{-1}B\sigma_1 = \\ & = B^*X^{-1}A_1^i[-A_1X - XA_1^*]X^{-1}B\sigma_1 = \\ & = -H_{i+1} - B^*X^{-1}A_1^iXA_1^*X^{-1}B\sigma_1 = \\ & = -H_{i+1} - B^*X^{-1}A_1^{i-1}[-B\sigma_1B^* - XA_1^*]A_1^*X^{-1}B\sigma_1 = \\ & = -H_{i+1} + B^*X^{-1}A_1^{i-1}B\sigma_1B^*A_1^*X^{-1}B\sigma_1 + \\ & +B^*X^{-1}A_1^{i-1}XA_1^*A_1^*X^{-1}B\sigma_1 = \\ & = -H_{i+1} + H_{i-1}\sigma_1^{-1}H_1^*\sigma_1 + B^*X^{-1}A_1^{i-1}XA_1^{2*}X^{-1}B\sigma_1 = \\ & = \cdots = \\ & = -H_{i+1} + H_{i-1}\sigma_1^{-1}H_1^*\sigma_1 - H_{i-2}\sigma_1^{-1}H_2\sigma_1 + \\ & + \cdots + (-1)^i\sigma_1^{-1}H_{i+1}^*\sigma_1 \end{array}$$

Consequently, the following formula holds:

$$H_{i+1}\sigma_1^{-1} + (-1)^i\sigma_1^{-1}H_{i+1}^* = \sum_{j=0}^{i}(-1)^{j+1}H_{i-j}\sigma_1^{-1}H_j^*. \tag{5.5}$$

Finally, we show how the third condition in Proposition 2.4 is reflected in the moments $H_i(t_2)$. This condition means that the function $S(\lambda, t_2)$ is $\sigma_1(t_2)$ contractive, and, for example, the first moment $H_0(t_2)$ satisfies $H_0(t_2)\sigma_1^{-1}(t_2) > 0$. Using the minimality property, we have

$$\operatorname{span} A^nB\mathbb{C}^p = \mathbb{C}^P,$$

where P is the dimension of the state space. Thus, in order to have $\mathbb{X} > 0$ (or $\mathbb{X}^{-1} > 0$) it is enough to demand

$$B^*(A^*)^n\mathbb{X}^{-1}A^nB > 0, \text{ for each } n.$$

Using the Lyapunov (3.8) equation iteratively we shall obtain that this condition becomes

$$\begin{array}{ll} H_0\sigma_1^{-1} > 0 & n = 0, \\ -H_2\sigma_1^{-1} - \sigma_1^{-1}H_0^*\sigma_1H_1\sigma_1^{-1} > 0 & n = 1, \\ H_4\sigma_1^{-1} + \sigma_1^{-1}H_0^*\sigma_1H_3\sigma_1^{-1} - \sigma_1^{-1}H_1^*\sigma_1H_2\sigma_1^{-1} > 0 & n = 2, \\ \ldots & \end{array}$$

More generally, for each n

$$(-1)^n H_{2n}\sigma_1^{-1} + \sum_{i=0}^{n-1}(-1)^{i+n}\sigma_1^{-1}H_i^*\sigma_1 H_{2n-1-i}\sigma_1^{-1} > 0. \tag{5.6}$$

Theorem 5.3. *Suppose that we are given moments $H_i(t_2)$, defined in a neighborhood of the point $t_2^0 \in \mathrm{I}$. Then there exists $n_0 \leq p^2$ such that each element H_{i+1} is uniquely determined from $H_0, \ldots, H_i$ using the algebraic formulas* (5.5)*, and n_0 LDEs with arbitrary initial conditions, obtained from* (5.4)*.*

Remarks: 1. This theorem is of local nature. The number "n_0" appearing in the theorem may vary with the point t_2^0, but is unchanged in a small neighborhood of t_2^0 by continuity.
2. We emphasize that in order to generate moments one needs produce n_0 initial conditions for each moment H_i. In other words, one need $2n_0$ complex numbers to determine H_0, H_1, etc.
3. The proof of the theorem allows to produce an algorithm to determine explicitly, up to n_0 initial conditions, the Markov moments. The arguments in the proof of the theorem are illustrated on an example in the following subsection. This example exhibits all the difficulties present in the general case.

Proof of 5.3: From formula (5.5) we obtain that the real or imaginary part of $H_{i+1}(t_2)$ is uniquely determined from all the previous moments. Suppose that $\Re H_{i+1}(t_2)$ is known (that is, i is even); then using formula (5.4) in which we insert the formula for $\Re H_{i+1}(t_2)$ we shall obtain an algebraic equation which gives $p^2 - n_0$ relations on the elements of $\Im H_{i+1}(t_2)$, where n_0 is the dimension of the kernel of the linear operator defined by the left handside of (5.4). Let us demonstrate it more explicitly, let us rewrite the equation (5.4) as a system of p^2 linear equations in p^2 variables $\Im H_{i+1}^{kl}$, $k, l = 1, \ldots, p^2$:

$$\left\{\begin{array}{l} \sum \alpha_{kl}^{11}\Im H_{i+1}^{kl} = \frac{d}{dt_2}H_i^{11} + \sum[\beta_{kl}^{11}H_i^{kl} + \delta_{kl}^{11}\Re H_{i+1}] \\ \sum \alpha_{kl}^{12}\Im H_{i+1}^{kl} = \frac{d}{dt_2}H_i^{12} + \sum[\beta_{kl}^{12}H_i^{kl} + \delta_{kl}^{12}\Re H_{i+1}] \\ \vdots \\ \sum \alpha_{kl}^{nn}\Im H_{i+1}^{kl} = \frac{d}{dt_2}H_i^{nn} + \sum[\beta_{kl}^{nn}H_i^{kl} + \delta_{kl}^{nn}\Re H_{i+1}] \end{array}\right. \tag{5.7}$$

Here $\alpha_{kl}^{ij}, \beta_{kl}^{ij}, \delta_{kl}^{ij}$ are functions of t_2 derived from the vessel parameters. Using basic algebra manipulations, it is enough to find a maximally independent subset of equations in the left side of (5.7), and to obtain

a system of $p^2 - n_0$ independent equations

(5.8)
$$\begin{cases} \sum_{kl} \alpha^1_{kl}(t_2)\Im H^{kl}_{i+1} = f_1(\frac{d}{dt_2}\Im H_i(t_2)^{kl}, \Im H_i(t_2), \Re H_{i+1}) \\ \sum_{kl} \alpha^2_{kl}(t_2)\Im H^{kl}_{i+1} = f_2(\frac{d}{dt_2}\Im H_i(t_2)^{kl}, \Im H_i(t_2), \Re H_{i+1}) \\ \vdots \\ \sum_{kl} \alpha^{p^2-n_0}_{kl}(t_2)\Im H^{kl}_{i+1} = f_{p^2-n_0}(\frac{d}{dt_2}\Im H_i(t_2)^{kl}, \Im H_i(t_2), \Re H_{i+1}) \end{cases}$$

for linear functions f_j and a system of linear dependent ones

$$\begin{cases} 0 = g_1(\frac{d}{dt_2}\Im H_i(t_2), \Im H_i(t_2), \Re H_{i+1}) \\ 0 = g_2(\frac{d}{dt_2}\Im H_i(t_2), \Im H_i(t_2), \Re H_{i+1}) \\ \vdots \\ 0 = g_{n_0}(\frac{d}{dt_2}\Im H_i(t_2), \Im H_i(t_2), \Re H_{i+1}) \end{cases}$$

for linear functions g_j.

Consequently, from (5.8) we express $p^2 - n_0$ elements of $\Im H_{i+1}$ as functions of other n_0 elements of H_{i+1} and of H_i. On the other hand, using the same formula (5.4) with $i+1$ plugged instead of i, we shall obtain n_0 differential equations for elements of $\Im H_{i+1}$, which can be considered as restrictions on n_0 unknown elements:

$$0 = g_j(\frac{d}{dt_2}\Im H_{i+1}(t_2), \Im H_{i+1}(t_2)), j = 1, \ldots, n_0. \tag{5.9}$$

Notice that there are at least n_0 unknown elements between these equations, because of the appearance of $\frac{d}{dt_2}\Im H_{i+1}(t_2)$ at the functions g_j.

Since we obtain algebraic (5.8) and differential (5.9) equations, they are all independent. So, if *all* elements of $\Im H_{i+1}$ appear at these equations we shall obtain that one can uniquely solve them up to n_0 initial conditions.

If this is not the case, some of the elements, say $p_0 \leq n_0$ will not appear in the equations (5.8) and (5.9). We sum up these consideration with the following description: we have obtained that $p^2 - p_0$ elements of $\Im H_{i+1}$ are determined from p_0 unknown elements of $\Im H_{i+1}$ and from H_i via p^2 equations (5.8) and (5.9).

By induction the same will hold for H_{i+2}. In other words, $p^2 - p_0$ elements of H_{i+2} are determined from p_0 unknown elements of H_{i+2} and from H_{i+1} via p^2 equations (5.8) and (5.9). This means that p_0

relations are obtained for the elements of H_{i+2}, which are actually equations on the elements of H_{i+1}. This produce differential equations on p_0 unknown elements of H_{i+1}. Notice that the unknown elements will appear only with one differentiation, since their derivatives did not appear at the previous stage. Thus H_{i+1} is uniquely determined up to n_0 initial conditions for some of its elements. □

Next theorem put some light on the connection between equations, which determine the transfer function $S(\lambda, t_2)$ and its moments. This theorem is similar to the property of a solution of a Riccati equation [Ze, theorem 2.1]

Theorem 5.4. *Suppose that $S(\lambda, t_2)$ is a continuous function of t_2 for each λ, meromorphic in λ for each t_2 and satisfies $S(\infty, t_2) = I$. Suppose also that $S(\lambda, t_2)$ is an intertwining function of LDEs* (2.16) *and* (2.17)*. Then if the equality*

$$S(\lambda, t_2) = \sigma_1^{-1}(t_2) S^{-1*}(-\bar{\lambda}, t_2) \sigma_1(t_2)$$

holds for t_2^0, then it holds for all t_2.

Proof: Since $S(\lambda, t_2)$ intertwines solutions of (2.16) and (2.17), then it satisfies the differential equation (2.22)

$$\frac{\partial}{\partial t_2} S(\lambda, t_2) = \sigma_1^{-1}(t_2)(\sigma_2(t_2)\lambda + \gamma_*(t_2)) S(\lambda, t_2) - \\ - S(\lambda, t_2) \sigma_1^{-1}(t_2)(\sigma_2(t_2)\lambda + \gamma(t_2)).$$

Consequently, using properties of γ_*, γ appearing in definition 1.1 we obtain that the function $\sigma_1^{-1}(t_2) S^{-1*}(-\bar{\lambda}, t_2) \sigma_1(t_2)$ satisfies the same differential equation. If these two functions are equal at t_2^0, from the uniqueness of solution for a differential equation with continuous coefficients, they are also equal for all t_2. □

Corollary 5.5. *Suppose that $S(\lambda, t_2)$ satisfies conditions of theorem 5.4, then the moments the equation for the moments* (5.5) *is excessive.*

Proof: From theorem 5.4 it follows that

$$S(\lambda, t_2) \sigma_1^{-1}(t_2) S^*(-\bar{\lambda}, t_2) \sigma_1(t_2) = I,$$

where taking the expansion in moments for $S(\lambda, t_2)$ and for $S^*(-\bar{\lambda}, t_2)$ we will get the formulas (5.5). □

We want to present next a necessary restriction on $\gamma_*(t_2)$, derived from the existence of a finite dimensional vessel:

Theorem 5.6. *Let $\sigma_1, \sigma_2, \gamma, \gamma_*$ be vessel parameters, and $\mathfrak{V}$ a finite dimensional vessel corresponding to the m parameters. Then the entreis of the function γ_* satisfies a polynomial differential equation of*

finite order with coefficients in the differential ring $\mathcal{R}$, generated by (the entries of) $\sigma_1, \sigma_1^{-1}, \sigma_2, \gamma$.

Proof: Suppose that the transfer function of the vessel $\mathfrak{V}$, defined in (2.11) is

$$S(\lambda, t_2) = I - B^*(t_2)\mathbb{X}^{-1}(t_2)(\lambda I - A_1)^{-1}B(t_2)\sigma_1(t_2) = I - \sum_{i=0}^{\infty} \frac{H_i}{\lambda^{i+1}},$$

and the linkage condition is

$$\sigma_1^{-1}(t_2)\gamma_*(t_2) = \sigma_1^{-1}(t_2)\gamma(t_2) + [\sigma_1^{-1}(t_2)\sigma_2(t_2), H_0(t_2)].$$

Notice that if we differentiate this formula and use the equation for the derivative of $H_0(t_2)$ from the equation (5.4), we shall get

$$\begin{aligned} \frac{d}{dt_2}\big(\sigma_1^{-1}(t_2)\gamma_*(t_2)\big) &= \frac{d}{dt_2}\big(\sigma_1^{-1}(t_2)\gamma(t_2)\big) + [\frac{d}{dt_2}(\sigma_1^{-1}(t_2)\sigma_2(t_2)), H_0(t_2)] + \\ &+ [\sigma_1^{-1}(t_2)\sigma_2(t_2), \frac{d}{dt_2}H_0(t_2)] = \\ &= \frac{d}{dt_2}\big(\sigma_1^{-1}(t_2)\gamma(t_2)\big) + f_{00}(H_0(t_2)) + f_{01}(H_1(t_2)) \end{aligned}$$

for linear in moments functions f_{00}, f_{01} with coefficients depending on $\mathcal{R}$ and γ_*. Similarly, differentiating this expression and using formula (5.4) for $\frac{d}{dt_2}H_0(t_2)$ and $\frac{d}{dt_2}H_1(t_2)$, we shall obtain that the second derivative is

$$\begin{aligned} &\frac{d^2}{dt_2^2}\big(\sigma_1^{-1}(t_2)\gamma_*(t_2)\big) = \\ &\quad = \frac{d^2}{dt_2^2}\big(\sigma_1^{-1}(t_2)\gamma(t_2)\big) + f_{10}(H_0) + f_{11}(H_1(t_2)) + f_{12}(H_2(t_2)). \end{aligned}$$

for linear in the moments functions f_{10}, f_{11}, f_{12} with coefficients depending on $\mathcal{R}$ and $\gamma_*, \frac{d}{dt_2}\gamma_*$. Continuing this differentiation further at each step i we shall obtain an equation of the form

$$\frac{d^i}{dt_2^i}\big(\sigma_1^{-1}(t_2)\gamma_*(t_2)\big) = \frac{d^i}{dt_2^i}\big(\sigma_1^{-1}(t_2)\gamma(t_2)\big) + \sum_{j=0}^{i} f_{ij}(H_j),$$

where f_{ij} is a linear function of H_j with coefficients, depending on $\mathcal{R}$ and first $i-1$ derivatives of γ_* (this can be immediately seen by the induction). But at some stage the moments start to repeat themselves

due to equation (5.1). So, taking K derivatives we shall obtain equations of the following form

(5.10)

$$\begin{cases} \sigma_1^{-1}(t_2)\gamma_*(t_2) = \sigma_1^{-1}(t_2)\gamma(t_2) + f_{00}(H_0(t_2)) \\ \frac{d}{dt_2}\big(\sigma_1^{-1}(t_2)\gamma_*(t_2)\big) = \frac{d}{dt_2}\big(\sigma_1^{-1}(t_2)\gamma(t_2)\big) + f_{10}(H_0(t_2)) + f_{11}(H_1(t_2)) \\ \frac{d^2}{dt_2^2}\big(\sigma_1^{-1}(t_2)\gamma_*(t_2)\big) = \frac{d^2}{dt_2^2}\big(\sigma_1^{-1}(t_2)\gamma(t_2)\big) + \sum_{j=0}^{2} f_{2j}(H_j) \\ \vdots \\ \frac{d^N}{dt_2^N}\big(\sigma_1^{-1}(t_2)\gamma_*(t_2)\big) = \frac{d^N}{dt_2^N}\big(\sigma_1^{-1}(t_2)\gamma(t_2)\big) + \sum_{j=0}^{N} f_{Nj}(H_j) \\ \vdots \\ \frac{d^K}{dt_2^K}\big(\sigma_1^{-1}(t_2)\gamma_*(t_2)\big) = \frac{d^K}{dt_2^K}\big(\sigma_1^{-1}(t_2)\gamma(t_2)\big) + \sum_{j=0}^{N} f_{Kj}(H_j) \end{cases}$$

Suppose that that the dimension of the inner state is n (i.e. $\dim \mathcal{H} = n$), then we get that each of the matrices H_j has n^2 entries and there is the total number of $n^2(N+1)$ entries for the moments $H_0, \dots, H_N$. So, taking "enough" derivatives of $\gamma_*(t_2)$ (i.e., taking K so that $K\dim(\mathcal{E})^2 > n^2(N+1)$) and eliminating all the entries of the moments, we shall obtain a finite number of polynomial differential equation for the entries of $\gamma_*(t_2)$. □

Remark: From this theorem is follows that γ_* satisfies an equation of the form

$$P(x, x', x'', \dots, x^{(K)} = 0,$$

where $P(x_0, x_1, x_2, \dots, x_K)$ is a non-commutative polynomial with coefficients in $\boldsymbol{\mathcal{R}}$.

5.2. **Sturm Liouville vessel parameters.** The following example was extensively studied in [M2]. It deals with the Sturm Liouville differential equation

$$\frac{d^2}{dt_2^2}y(t_2) - q(t_2)y(t_2) = \lambda y(t_2),$$

with the spectral parameter λ. The parameter $q(t_2)$ is usually called the *potential*. For $q(t_2) = 0$ this problem is easily solved by exponents and in this case we shall call this equation *trivial*. In [M2] one connects solutions of the more general problem with non trivial $q(t_2)$ to the trivial one.

Definition 5.7. *the Sturm Liouville vessel parameters are given by*

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \sigma_2 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \gamma = \begin{bmatrix} 0 & 0 \\ 0 & i \end{bmatrix},$$
$$\gamma_*(t_2) = \begin{bmatrix} -i\pi_{11}(t_2) & -\beta(t_2) \\ \beta(t_2) & i \end{bmatrix}$$

for ***real*** *valued continuous functions $\pi_{11}(t_2), \beta(t_2)$.*

The input compatibility differential equation (2.16) is equivalent to

$$\begin{cases} \lambda u_1(\lambda, t_2) - \frac{\partial}{\partial t_2} u_2(\lambda, t_2) = 0 \\ -\frac{\partial}{\partial t_2} u_1(\lambda, t_2) + i u_2(\lambda, t_2) = 0 \end{cases}$$

where we denote $u_\lambda(t_2) = \begin{bmatrix} u_1(\lambda, t_2) \\ u_2(\lambda, t_2) \end{bmatrix}$. From the second equation one finds that $u_2(\lambda, t_2) = -i\frac{\partial}{\partial t_2} u_1(\lambda, t_2)$ and plugging it back to the first equation, we shall obtain the trivial Sturm Liouville differential equation with the spectral parameter $i\lambda$ for $u_1(\lambda, t_2)$:

$$\frac{\partial^2}{\partial t_2^2} u_1(\lambda, t_2) = i\lambda u_1(\lambda, t_2).$$

For the output $y_\lambda(t_2) = \begin{bmatrix} y_1(\lambda, t_2) \\ y_2(\lambda, t_2) \end{bmatrix}$, we shall obtain that (2.17) is equivalent to the system of equations

$$\begin{cases} (\lambda - i\pi_{11}(t_2)) y_1(\lambda, t_2) - (\frac{\partial}{\partial t_2} + \beta(t_2)) y_2(\lambda, t_2) = 0 \\ (\beta(t_2) - \frac{\partial}{\partial t_2}) y_1(\lambda, t_2) + i y_2(\lambda, t_2) = 0 \end{cases},$$

from which we immediately obtain that $y_2(\lambda, t_2) = i(\beta(t_2) - \frac{\partial}{\partial t_2}) y_1(\lambda, t_2)$ and plugging it into the first equation

$$\frac{\partial^2}{\partial t_2^2} y_1(\lambda, t_2) - (\pi_{11}(t_2) + \beta'(t_2) + \beta^2(t_2)) y_1(\lambda, t_2) = i\lambda y_1(\lambda, t_2),$$

which means that $y_1(\lambda, t_2)$ satisfies the Sturm Liouville differential equation with the spectral parameter $i\lambda$ and the potential $q(t_2) = (\pi_{11}(t_2) + \beta'(t_2) + \beta^2(t_2))$.

The first equation (1.2) considered for $H_0 = \begin{bmatrix} H_0^{11} & H_0^{12} \\ H_0^{21} & H_0^{22} \end{bmatrix}$ becomes

$$\begin{bmatrix} -i\pi_{11}(t_2) & -\beta(t_2) \\ \beta(t_2) & 0 \end{bmatrix} = \begin{bmatrix} H_0^{11} - H_0^{22} & H_0^{12} \\ -H_0^{12} & 0 \end{bmatrix}.$$

from where we conclude that

$$H_0^{12} = -\beta(t_2), \quad H_0^{11} - H_0^{22} = -i\pi_{11}(t_2). \tag{5.11}$$

Let us consider the differential equation (5.4), where we use the notation $H_1 = \begin{bmatrix} H_1^{11} & H_1^{12} \\ H_1^{21} & H_1^{22} \end{bmatrix}$. Substituting the expressions for the vessel parameters, we shall obtain

$$\begin{cases} \frac{d}{dt_2} H_0^{11} - \beta H_0^{11} - i H_0^{21} & = -H_1^{12}, \\ \frac{d}{dt_2} H_0^{12} - \beta H_0^{12} + i(H_0^{11} - H_0^{22}) & = 0, \\ \frac{d}{dt_2} H_0^{21} + i\pi_{11} H_0^{11} + \beta H_0^{21} & = H_1^{11} - H_1^{22}, \\ \frac{d}{dt_2} H_0^{22} + i\pi_{11} H_0^{12} + \beta H_0^{22} + i H_0^{21} & = H_1^{12}. \end{cases}$$

and consequently, using the formulas (5.11) the second equation results in

$$\pi_{11} = \frac{d}{dt_2}\beta - \beta^2. \tag{5.12}$$

Together the first and the fourth equations give

$$H_1^{12} = -(\frac{d}{dt_2} H_0^{11} - \beta H_0^{11} - i H_0^{21}) = \frac{d}{dt_2} H_0^{22} + i\pi_{11} H_0^{12} + \beta H_0^{22} + i H_0^{21}$$

from where we obtain using (5.11)

$$\frac{d}{dt_2}[H_0^{11} + H_0^{22}] = 0 \Rightarrow H_0^{11} + H_0^{22} = C \in \mathbb{C}, \tag{5.13}$$

Additionally, H_0 has to satisfy $H_0 = \sigma_1^{-1} H_0^* \sigma_1$. Using this relation we shall obtain that

$$\begin{bmatrix} H_0^{11} & H_0^{12} \\ H_0^{21} & H_0^{22} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} (H_0^{11})^* & (H_0^{21})^* \\ (H_0^{12})^* & (H_0^{22})^* \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \Rightarrow$$
$$\Rightarrow \begin{cases} H_0^{11} = (H_0^{22})^* \\ H_0^{12} = (H_0^{12})^* \\ H_0^{21} = (H_0^{21})^* \\ H_0^{22} = (H_0^{11})^* \end{cases}$$

from where we conclude that

$$H_0^{11} = (H_0^{22})^*, \quad H_0^{12} = (H_0^{12})^*, \quad H_0^{21} = (H_0^{21})^*. \tag{5.14}$$

As we can see $H_0^{21} = h_0^{21}$ is a real valued (arbitrary at this stage) function, and H_0 is as follows

$$H_0 = \begin{bmatrix} \dfrac{r - i\pi_{11}}{2} & -\beta \\ h_0^{21} & \dfrac{r + i\pi_{11}}{2} \end{bmatrix}, \tag{5.15}$$

where $r \in \mathbb{R}$.

Let us perform the same calculations for H_1. From the algebraic equation (5.5)

$$H_1\sigma_1^{-1} + \sigma_1^{-1}H_1^* = -H_0\sigma_1^{-1}H_0^*$$

we obtain that

$$\begin{bmatrix} H_1^{12} + (H_1^{12})^* & H_1^{11} + (H_1^{22})^* \\ H_1^{22} + (H_1^{11})^* & H_1^{21} + (H_1^{21})^* \end{bmatrix} = -H_0\sigma_1^{-1}H_0^*$$

From the equation (5.4) with $i = 0$ we obtain as before that

$$H_1^{11} - H_1^{22} = \frac{d}{dt_2}H_0^{21} + i\pi_{11}H_0^{11} + \beta H_0^{21}, \tag{5.16}$$

$$H_1^{12} = \frac{d}{dt_2}H_0^{22} + i\pi_{11}H_0^{12} + \beta H_0^{22} + iH_0^{21}. \tag{5.17}$$

and the same equation (5.4) with $i = 1$ produces similarly to the previous case

$$\frac{d}{dt_2}H_1^{12} - \beta H_1^{12} + i(H_1^{11} - H_1^{22}) = 0, \tag{5.18}$$

$$\frac{d}{dt_2}(H_1^{11} + H_1^{22}) = -i\pi_{11}H_1^{12} + \beta(H_1^{11} - H_1^{22}). \tag{5.19}$$

Plugging (5.16) and (5.17) into (5.18), we shall obtain that $H_0^{21} = h_0^{21}$ have to satisfy the following differential equation of the first order:

$$\begin{aligned} &\frac{d}{dt_2}(\tfrac{d}{dt_2}H_0^{22} + i\pi_{11}H_0^{12} + \beta H_0^{22} + iH_0^{21}) - \\ &-\beta(\tfrac{d}{dt_2}H_0^{22} + i\pi_{11}H_0^{12} + \beta H_0^{22} + iH_0^{21}) + \\ &+i\tfrac{d}{dt_2}H_0^{21} - \pi_{11}H_0^{11} + i\beta H_0^{21} = 0. \end{aligned}$$

or after cancellations

$$2i(H_0^{21})' = i(\pi_{11}H_0^{12})' + +\beta\pi_{11}H_0^{12} - \pi_{11}(H_0^{22} - H_0^{11}) - \frac{d^2}{dt_2^2}H_0^{22}. \tag{5.20}$$

Inserting here the expressions for H_0^{ij} appearing in (5.15) we shall obtain that the real part of the last equality can be derived from the equation (5.12):

$$\frac{r}{2}[\beta' - \beta^2 - \pi_{11}] = 0.$$

The imaginary part gives the following equation

$$4\frac{d}{dt_2}h_0^{21} = \frac{d}{dt_2}(\pi_{11}\beta) + \beta\pi_{11}' - \beta^2\pi_{11} - \pi_{11}^2 - \pi_{11}''.$$

Suppose (see [M2]) that there exists a function τ such that $\beta = -\dfrac{\tau'}{\tau}$. Then using (5.12) $\pi_{11} = -\dfrac{\tau''}{\tau}$ and inserting these equations into the

formula for $\frac{d}{dt_2}h_0^{21}$ we obtain that

$$4\frac{d}{dt_2}h_0^{21} = \frac{\tau^{(IV)}}{\tau} - (\frac{\tau''}{\tau})^2.$$

Let us write down the formulas for the elements of H_1:

$$\begin{cases} H_1^{12} = \frac{d}{dt_2}H_0^{22} + i\pi_{11}H_0^{12} + \beta H_0^{22} + iH_0^{21}, & (5.17) \\ H_1^{11} - H_1^{22} = \frac{d}{dt_2}H_0^{21} + i\pi_{11}H_0^{11} + \beta H_0^{21}, & (5.16) \\ \frac{d}{dt_2}(H_1^{11} + H_1^{22}) = -i\pi_{11}H_1^{12} + \beta(H_1^{11} - H_1^{22}), & (5.19) \\ 2i(H_1^{21})' = i(\pi_{11}H_1^{12})' + \beta\pi_{11}H_1^{12} - \pi_{11}(H_1^{22} - H_1^{11}) - \frac{d^2}{dt_2^2}H_1^{22} & (5.20)' \end{cases}$$

The last equation $(5.20)'$ is obtained from (5.20) by substituting the index 0 at H_0^{ij} by the index 1: H_1^{ij}.
Finally, we obtain that in the general case H_{i+1} is derived from H_i using a system of similar equations. Denote

$$\begin{aligned} M_i &= \tfrac{d}{dt_2}H_i^{22} + i\pi_{11}H_i^{12} + \beta H_i^{22} + iH_i^{21}, \\ L_i &= \tfrac{d}{dt_2}H_i^{21} + i\pi_{11}H_i^{11} + \beta H_i^{21} \end{aligned}$$

then
(5.21)

$$\begin{cases} H_{i+1}^{12} & = M_i, \\ H_{i+1}^{11} - H_{i+1}^{22} & = L_i, \\ \frac{d}{dt_2}(H_{i+1}^{11} + H_{i+1}^{22}) & = -i\pi_{11}M_i + \beta L_i, \\ 2i\frac{d}{dt_2}H_{i+1}^{21} & = -\dfrac{L_i''}{2} + \dfrac{i}{2}(\pi_{11}M_i)' + \dfrac{(\beta L_i)'}{2} + \beta\pi_{11}M_i + \pi_{11}L_i. \end{cases}$$

from where we see that Markov moments are defined up to initial conditions for $n_0 = 2$ elements. Notice that $p_0 = 1$ in this case.
Let us also demonstrate theorem 5.6 for the Sturm Liouville parameters from definition 5.7. We will take the simplest case $n = 1$ and as a result the transfer function is

$$S(\lambda, t_2) = I - \frac{1}{\lambda + z}C(t_2)B(t_2)\sigma_1 = I - \sum_{i=0}^{\infty} \frac{1}{\lambda^{i+1}}(-z)^i C(t_2)B(t_2)\sigma_1.$$

We have already seen (in (5.12)) that γ_* is necessarily of the form

$$\gamma_*(t_2) = \begin{bmatrix} i(\beta' - \beta^2) & -\beta \\ \beta & i \end{bmatrix}$$

for a real valued function $\beta(t_2)$ on I. In this case, $N = 0$ which means that the first moment is a multiple of the zero moment: $H_1 = -zH_0$. Since the vessel parameters are constant the differential ring $\boldsymbol{\mathcal{R}} = \mathbb{C}$

is trivial. Using the formulas developed in [M2] we shall obtain that $\tau = \exp \int \beta$ satisfies

$$(\frac{d}{dt_2} - k)(\frac{d}{dt_2} - \bar{k})(\frac{d}{dt_2} + k)(\frac{d}{dt_2} + \bar{k})\tau = 0.$$

for $k = \sqrt{-iz}$, which may obviously be rewritten as a polynomial differential equation for β, after inserting the formula for $\tau = \exp \int \beta$ and multiplying by τ^{-1}.

5.3. **Partial realization problem.** In this section we want to solve the partial realization problem 1.4. The key idea here is analyzing of the equations (1.2) and (5.4). First, from the equation (1.2) we find that

$$\gamma_*(t_2) = \gamma(t_2) + \sigma_2(t_2)H_0(t_2) - \sigma_1 H_0(t_2)\sigma_1^{-1}\sigma_2(t_2)$$

Consider now the system of equations (5.4) for $i = 1, \dots, n-1$, where we substitute $\gamma_*(t_2)$ by the formula (1.2) above:

$$\begin{array}{l}
\frac{d}{dt_2}H_i - \sigma_1^{-1}\gamma_* H_i + H_i\sigma_1^{-1}\gamma = \sigma_1^{-1}\sigma_2 H_{i+1} - H_{i+1}\sigma_1^{-1}\sigma_2 \Leftrightarrow \\
\frac{d}{dt_2}H_i - \sigma_1^{-1}(\gamma + \sigma_2 H_0 - \sigma_1 H_0\sigma_1^{-1}\sigma_2)H_i + H_i\sigma_1^{-1}\gamma = \\
= \sigma_1^{-1}\sigma_2 H_{i+1} - H_{i+1}\sigma_1^{-1}\sigma_2 \Leftrightarrow \\
-\sigma_1^{-1}\gamma H_i + H_i\sigma_1^{-1}\gamma = \\
= -\frac{d}{dt_2}H_i + (\sigma_2 H_0 - \sigma_1 H_0 \sigma_1^{-1}\sigma_2)H_i + \sigma_1^{-1}\sigma_2 H_{i+1} - H_{i+1}\sigma_1^{-1}\sigma_2
\end{array}$$

This condition can be rewritten in a more compact way as

$$[H_i, \sigma_1^{-1}\gamma] = -\frac{d}{dt_2}H_i - \sigma_1[H_0, \sigma_1^{-1}\sigma_2]H_i - [H_{i+1}, \sigma_1^{-1}\sigma_2], \tag{5.22}$$

where $[A, B] = AB - BA$ denotes the commutator of A and B.
So, the question which arises here is whether every function γ, which solves (5.22) produces a solution of the partial realization problem 1.4. We connect the solution of the partial realization problem 1.4 to the classical one.

Theorem 5.8. *Assume that we are given functions σ_1, σ_2 and Markov moments $H_i(t_2), i = 0, \dots, n$ ($n \geq 0$) defined in a neighborhood of a point $t_2^0 \in$ I, satisfying the necessary conditions* (5.5) *and* (5.6)*. Assume also that the following necessary conditions are also satisfied:*

(1) *The classical partial realization problem for the moments $H_i(t_2^0)$ $i = 1, \dots, n$ has a solution, which we denote by $S \in \mathcal{RS}$ with a minimal realization*

$$S(\lambda, t_2^0) = I - B_0\mathbb{X}_0^{-1}(\lambda I - A_0)^{-1}B_0\sigma_1(t_2^0)$$

(2) *There exists a solution $\gamma(t_2)$ of the equations* (5.22) *for each $i \in \{0, \dots, n\}$, and such that $B(t_2)$, $\mathbb{X}(t_2)$ defined from the formulas* (4.1), (4.3) *using B_0, A_0, $\mathbb{X}_0$ and $\gamma(t_2)$ realize the first moment:*

$$H_0(t_2) = (B(t_2))^* \mathbb{X}(t_2)^{-1} B(t_2) \sigma_1(t_2).$$

Then, the function

$$S(\lambda, t_2) = I - B(t_2) \mathbb{X}(t_2)^{-1} (\lambda I - A_0)^{-1} B(t_2) \sigma_1(t_2)$$

is a solution of the partial realization problem 1.4, with $\gamma_*(t_2)$ given by the equation (1.2).

Proof: Using simple calculation and the definitions of $B(t_2), \mathbb{X}(t_2)$ it follows that the function $S(\lambda, t_2)$ intertwines solutions of the input and output LDEs, with γ_* defined as in the theorem, is equal to I_p at ∞, and is continuous for each λ. Since the Lyapunov equation holds, the function $S(\lambda, t_2) \in \boldsymbol{\mathcal{RSI}}$. It remains to show that it has the given Markov moments. By construction the first moment of $S(\lambda, t_2)$ is $H_0(t_2)$. By Theorem 5.3 the second moment of $S(\lambda, t_2)$ satisfies (5.4) with $i = 0$, and differential equations with n_0 initial conditions. But these initial conditions are obtained from the realization $S(\lambda, t_2^0)$. This way we obtain the first moment of $S(\lambda, t_2)$. The other moments are obtained by iteration. $\square$

Remarks: 1. If there is a linear combination $H(t_2) = \sum\limits_{i=0}^{n-1} \alpha_i H_i(t_2)$ which has its spectrum disjoint from $-H(t_2)$ then γ is uniquely determined from

$$\begin{aligned} &[\sum_{i=0}^{n-1} \alpha_i H_i, \sigma_1^{-1}\gamma] = \\ &= -\frac{d}{dt_2} \sum_{i=0}^{n-1} \alpha_i H_i - \sigma_1 [H_0, \sigma_1^{-1}\sigma_2] \sum_{i=0}^{n-1} \alpha_i H_i - [\sum_{i=0}^{n-1} \alpha_i H_{i+1}, \sigma_1^{-1}\sigma_2] \end{aligned}.$$

2. Case $\sigma_1(t_2) > 0$: The equation (5.22) is then uniquely solvable for γ if the spectrum because the first Markov moment $H_0(t_2)$ is self adjoint and strictly positive, and thus the spectra of H_0 and $-H_0$ are disjoint.

6. Nevanlinna-Pick interpolation problem

Nevanlinna-Pick (NP) interpolation problem 1.6 is similar to non commutative, Riemann surface cases and involves specifying the exact class of input-output mappings, and a finite number of inputs that are to be mapped to the corresponding outputs. In our case this yields fixing

both $\mathbf{U}(\lambda, \dfrac{\partial}{\partial t_2}; t_2)$ and $\mathbf{U}_*(\lambda, \dfrac{\partial}{\partial t_2}; t_2)$, a set of values $w_i, i = 1, \ldots, N$ for the spectral parameter, and corresponding solutions $\xi_i(t_2)$ and $\eta_i(t_2)$ of the input and the output LDEs, respectively, as it is defined in the problem 1.6.

Since everything is determined by the initial conditions, this reduces to the NP problem at some fixed $t_2^0 \in \mathrm{I}$, except that now γ_* is fixed, yielding a constraint on the solution parameter S_0. According to remark, following the theorem 5.6 γ_* satisfies an equation of the form

$$P(x, x', x'', \ldots, x^{(K)}) = 0$$

for a non-commutative polynomial with coefficients in $\boldsymbol{\mathcal{R}}$. Since the derivatives of γ_* may be represented using the moments $H_i(t_2)$ differentiating the linkage condition (2.8) and using (5.4), we shall obtain that the moments have to satisfy an equation of the form

$$P'(H_0, H_1, \ldots, H_n) = 0$$

for a non-commutative polynomial P' with coefficients in $\boldsymbol{\mathcal{R}}$. On the other hand, they must be linearly dependent, which is a necessary condition and the formula (5.1) must hold. Consequently, there must be a minimal (with respect to the index N - the biggest moment appearing) linear combination

$$\sum_{i=1}^{N} \mu_i H_i(t_2)$$

of the moments such that it divides from the left and from the right the polynomial $P'(H_0, H_1, \ldots, H_n)$. In this way we shall be obtain the coefficients μ_i, which are also coefficients of the minimal polynomial for the final operator A_1.

Next we shall look for functions $S_0 \in \boldsymbol{\mathcal{RS}}$, which has A_1 with the prescribed minimal polynomial. Notice that this also means that the Jordan block structure of the class of A_1 is defined in this manner. It is remained to choose values for $B_0, \mathbb{X}_0$ so that the system of equations (5.10) will be satisfied at the point t_2^0. This follows from the fact that $\gamma_{*,0}$ corresponding to the function S_0 will have the same initial values as the given γ_* and, moreover, there will be obtained the same differential equations for the γ_* and $\gamma_{*,0}$. We now present the exact statement

Theorem 6.1. *Suppose that there is a realization of $S_0 \in \boldsymbol{\mathcal{RS}}$*

$$S_0(\lambda, t_2) = I - B_0^* \mathbb{X}^{-1} (\lambda I - A_1)^{-1} B_0 \sigma_1(t_2^0),$$

then S_0 is a solution of the Nevanlinna-Pick interpolation problem 1.6, if the minimal polynomial of A_1 is equal to the minimal polynomial

defined from γ_* and if the moments H_i of the functions S_0 satisfy the system of equations (5.10) *at t_2^0.*

Proof: We construct first a polynomial $P(x, x', x'', \ldots, x^{(K)})$ from γ_* as it is explained in the text preceding the theorem. Then we find a minimal polynomial for A_1, obtained from $P'(H_0, H_1, \ldots, H_n)$ by substituting H_i with x^i. Then the moments of the function $S_0(\lambda, t_2)\boldsymbol{\mathcal{RSI}}$ constructed from $S_0 \in \boldsymbol{\mathcal{RS}}$ will satisfy the same differential equations appearing in theorem (5.10). As a result the output $\gamma_{*,0}$ of the function $S_0(\lambda, t_2)$ will satisfy the same polynomial differential equation as the given γ_* and will have the same initial conditions, which means that $\gamma_{*,0} = \gamma_*$, which finishes the proof. $\square$

We now solve Problem 1.7.

Theorem 6.2. *Given $\mathbb{C}^{p\times p}$-valued functions σ_1, σ_2, γ, an interval* I *and N quadruples $< t_2^j, w_j, \xi_j, \eta_j >$ where $t_2^j \in \mathrm{I}$, $w_j \in \mathbb{C}_+$, $\xi_j, \eta_j \in \mathbb{C}^{1\times p}$ $j = 1, \ldots N$, and assume that the corresponding matrices $\widetilde{\mathbb{X}}_i > 0$. Then there exists a solution of the Nevanlinna-Pick problem 1.7, i.e. there exists a function $S \in \boldsymbol{\mathcal{RSI}}$ satisfying $S(w_i, t_2^i)\xi_i = \eta_i$ if and only if there exists $n \in \mathbb{N}$, matrices $A_0^i, \mathbb{X}_0^i, \in \mathbb{C}^{(n-1)\times(n-1)}$ with $\mathbb{X}_0^i > 0$, $B_0^i \in \mathbb{C}^{p\times(n-1)}, V_{ij} \in \mathbb{C}^{n\times n}$ such that for $A_i, B_i, \mathbb{X}_i$ defined by*

$$B_i = \begin{bmatrix} B_0^i \\ \eta_i - \xi_i \end{bmatrix} \tag{6.1}$$

$$\mathbb{X}_i = \begin{bmatrix} \mathbb{X}_0^i & 0 \\ 0 & \widetilde{\mathbb{X}}_i \end{bmatrix} \tag{6.2}$$

$$A_i = \begin{bmatrix} A_0^i & \dfrac{B_0^i\sigma_1(t_2^i)\xi_i^*}{\widetilde{\mathbb{X}}_i} \\ -\eta_i\sigma_1(t_2^i)(B_0^i)^*(\mathbb{X}_0^i)^{-1} & -w_i^* - \dfrac{\eta_i\sigma_1(t_2^i)(\eta_i^* - \xi_i^*)}{\widetilde{\mathbb{X}}_i} \end{bmatrix} \tag{6.3}$$

it holds that

1. $A_i = V_{ij}A_jV_{ij}^{-1}$,
2. $\oint(\lambda I - A_i)^{-1}B_i\sigma_1^{-1}(t_2^j)\Phi^{-1}(\lambda, t_2^j, t_2^i)d\lambda = V_{ij}B_j$.

and the matrix $X(t_2)$,

$$\mathbb{X}(t_2) = \mathbb{X}_i + \int_{t_2^i}^{t_2} B_i(y)\sigma_2(y)(B_i(y))^*dy$$

is invertible on the interval I.

Proof: For each t_2^i all the functions which satisfy $S(w_i, t_2^i)\xi_i = \eta_i$ are of the form $T_{\Theta_i}(S_0^i(\lambda, t_2))$, provided $\widetilde{\mathbb{X}}_i = \dfrac{\xi_i\sigma_1(t_2^i)\xi_i^* - \eta_i\sigma_1(t_2^i)\eta_i^*}{w_i^* + w_i} > 0$.

Here

$$\Theta_i(\lambda) = \begin{bmatrix} I + \dfrac{\eta_i^*\eta_i\sigma_1}{\widetilde{\mathbb{X}}_i(\lambda + w_i^*)} & \dfrac{\eta_i^*\xi_i\sigma_1}{\widetilde{\mathbb{X}}_i(\lambda + w_i^*)} \\ -\dfrac{\xi_i^*\eta_i\sigma_1}{\widetilde{\mathbb{X}}_i(\lambda + w_i^*)} & I - \dfrac{\xi^*\xi\sigma_1}{\mathbb{X}(\lambda + w^*)} \end{bmatrix}$$

and $S_0^i \in \boldsymbol{\mathcal{RS}}$. Given a minimal realization

$$S_0^i(\lambda) = I - (B_0^i)^*(\mathbb{X}_0^i)^{-1}(\lambda I - A_0^i)^{-1}B_0^i\sigma_1(t_2^i)$$

of S_0^i, we shall obtain from formulas (3.15), (3.16), (3.17) the formulas (6.1), (6.2), (6.3) in the theorem. In view of Theorem 4.7, a necessary and sufficient condition to obtain the same function $S(\lambda, t_2)$ for every i is that there exist invertible constant matrices V_{ij} such that the operators A_i are similar. In other words there must exist V_{ij} such that $A_i = V_{ij}A_jV_{ij}^{-1}$. Moreover, the second part of theorem 4.7 tells that additionally the equality

$$\oint (\lambda I - A_i)^{-1}B_i\sigma_1^{-1}(t_2^j)\Phi^{-1}(\lambda, t_2^j, t_2^i)d\lambda = V_{ij}B_j$$

must hold. □

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ON THE CLASS $\mathcal{SI}$ OF J-CONTRACTIVE FUNCTIONS INTERTWINING SOLUTIONS OF LINEAR DIFFERENTIAL EQUATIONS

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Abstract. In the PhD thesis of the second author under the supervision of the third author was defined the class $\mathcal{SI}$ of J-contractive functions, depending on a parameter and arising as transfer functions of overdetermined conservative $2D$ systems invariant in one direction. In this paper we extend and solve in the class $\mathcal{SI}$, a number of problems originally set for the class $\mathcal{S}$ of functions contractive in the open right-half plane, and unitary on the imaginary line with respect to some preassigned signature matrix J. The problems we consider include the Schur algorithm, the partial realization problem and the Nevanlinna-Pick interpolation problem. The arguments rely on a correspondence between elements in a given subclass of $\mathcal{SI}$ and elements in $\mathcal{S}$. Another important tool in the arguments is a new result pertaining to the classical tangential Schur algorithm.

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1991 *Mathematics Subject Classification.* Primary:47A48, 93C15, 93C35; Secondary: 16E45, 46E22.

Key words and phrases. Schur algorithm, time-invariant $2D$-systems, linear differential equations.

D. Alpay thanks the Earl Katz family for endowing the chair which supported his research. The research of the authors was supported in part by the Israel Science Foundation grant 1023/07.

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1. Introduction

Functions $S(\lambda)$, which are $\mathbb{C}^{p\times p}$-valued, analytic and contractive in the open right half plane $\mathbb{C}_+$, or equivalently, such that the kernel

$$K(\lambda, w) = \frac{I_p - S(w)^* S(\lambda)}{\lambda + w^*}$$

is positive[1] in $\mathbb{C}_+$, play an important role in system theory, inverse scattering theory, network theory and related topics; see for instance [L], [BC], [DD], [He], [A]. Here, positivity of the kernel means that for every $n \in \mathbb{N}$ and every choice of points $w_1, \ldots, w_n \in \mathbb{C}_+$ and vectors $\xi_1, \ldots, \xi_n \in \mathbb{C}^{1\times p}$ the $n \times n$ Hermitian matrix

$$\left[\xi_i K(w_i, w_j)\xi_j^*\right]_{i,j=1,\ldots n}$$

is positive (that is, has all its eigenvalues greater or equal to 0).

Far reaching generalizations of this class were introduced in [M, MV1, MVc], in the study of $2D$-linear systems (say, with respect to the variables (t_1, t_2)), invariant with respect to the variable t_1. To introduce the classes defined in these papers we first need a definition.

Definition 1.1. *Let σ_1, σ_2, γ and γ_* be $\mathbb{C}^{p\times p}$-valued functions, continuous on an interval $\mathrm{I} = [a, b]$. Suppose moreover that σ_1 and σ_2 take self-adjoint values, and that σ_1 is differentiable and invertible on I, and that the following relations hold:*

$$\gamma(t_2) + \gamma(t_2)^* = \gamma_*(t_2) + \gamma_*(t_2)^* = -\frac{d}{dt_2}\sigma_1(t_2), \quad t_2 \in \mathrm{I}.$$

[1]positive-definite in the classical terminology

Then $\sigma_1, \sigma_2, \gamma, \gamma_*$ and the interval I *are called* ***vessel parameters****.*

The class of functions $\mathcal{SI}$ corresponding to thr vessel parameters σ_1, σ_2, γ, γ_* and the interval I was introduced in [M, MVc] (see Definition 2.7 below) and consists of the functions $S(\lambda, t_2)$ of two variables λ, t_2 such that for every $t_2 \in \mathrm{I}$ the function $S(\lambda, t_2)$ is meromorphic in $\mathbb{C}_+$ and the kernel

$$\frac{\sigma_1(t_2) - S(w, t_2)^* \sigma_1(t_2) S(\lambda, t_2)}{\lambda + w^*} \tag{1.1}$$

is positive for λ and ω in the domain of analyticity of $S(\lambda, t_2)$ in $\mathbb{C}_+$. For positive $\sigma_1(t_2)$, the positivity of the kernel implies that S is analytic in $\mathbb{C}_+$; see [Do], [A]. For general (invertible) $\sigma_1(t_2)$, the entries of S are of bounded type and S has at most poles in $\mathbb{C}_+$; see [ADRS]. It is also required that $S(\lambda, t_2)$ is analytic at infinity for each t_2, with value I_p there, and that $S(\lambda, t_2)$ maps solutions of the *input* Linear Differential Equation (LDE) with the spectral parameter λ

$$\lambda \sigma_2(t_2) u(\lambda, t_2) - \sigma_1(t_2) \frac{\partial}{\partial t_2} u(\lambda, t_2) + \gamma(t_2) u(\lambda, t_2) = 0,$$

to solutions of the *output* LDE

$$\lambda \sigma_2(t_2) y(\lambda, t_2) - \sigma_1(t_2) \frac{\partial}{\partial t_2} y(\lambda, t_2) + \gamma_*(t_2) y(\lambda, t_2) = 0.$$

It is proved in [M, MVc] that elements of $\mathcal{SI}$ are the transfer functions of t_1-invariant conservative $2D$ systems; see Section 2.

The purpose of this paper is to study various questions in the class $\mathcal{SI}$ such as Nevanlinna-Pick interpolation, property of moments, etc. A key result is the following theorem, which we prove in the sequel; see Section 4.

Theorem 4.1 *Let us fix the parameters σ_1, σ_2, and γ, and the interval* I*. Then for every $t_2^0 \in \mathrm{I}$ there is a one-to-one correspondence between pairs (γ_*, S) such that $S \in \mathcal{SI}$ and γ_* continuous in a neighborhood of t_2^0, and functions $Y(\lambda)$, meromorphic in $\mathbb{C}_+$, and with the following properties*

1. $Y(\infty) = I_p$,
2. $Y(\lambda)^* \sigma_1(t_2^0) Y(\lambda) \leq \sigma_1(t_2^0)$ *for $\lambda \in \mathbb{C}_+$ where Y is analytic, and*
3. $Y(\lambda)^* \sigma_1(t_2^0) Y(\lambda) = \sigma_1(t_2^0)$ *for almost λ satisfying $\Re\lambda = 0$ and where $Y(\lambda)$ is the non-tangential limit.*

As mentioned above the $\sigma_1(t_2)$-contractivity of Y implies that Y is of bounded type in $\mathbb{C}_+$, and thus the asserted non-tangential limits exist almost everywhere.

Definition 1.2. *The class of functions Y with the properties in Theorem 4.1 will be will be denoted by $\boldsymbol{\mathcal{S}}(t_2^0)$ and the functions will be called $\sigma_1(t_2^0)$-inner. The subclass of rational functions of $\boldsymbol{\mathcal{S}}(t_2^0)$ will be denoted by $\boldsymbol{\mathcal{RS}}(t_2^0)$.*

For the sequel, it is important to notice that the general tangential Schur algorithm developped in [AD] can be applied to functions in $\boldsymbol{\mathcal{S}}(t_2^0)$, and in particular in $\boldsymbol{\mathcal{RS}}(t_2^0)$.

Definition 1.3. *$\boldsymbol{\mathcal{RSI}}$ will denote the subclass of functions in $\boldsymbol{\mathcal{SI}}$ which are rational in λ for every $t_2 \in \mathrm{I}$.*

The paper consists of six sections besides the introduction, and we now describe their content. In Section 2 we review the main results from [M] (see details in [MV1] and [MVc]) on t_1-invariant conservative $2D$-systems, relevant to the present work. In particular the class $\boldsymbol{\mathcal{SI}}$ mentioned above consists of the transfer functions of these systems. In Section 3 we present the reproducing kernel space approach to the tangential Schur algorithm for the class $\boldsymbol{\mathcal{S}}$, as developed in [AD]. We obtain in particular new formulas which allow us to find the main operator in a realization of an element of $\boldsymbol{\mathcal{S}}$ after one iteration of the tangential Schur algorithm; see formulas (3.15), (3.17), (3.16) in Theorem 3.5. In Section 4 we develop the tangential Schur algorithm for a function $S(\lambda, t_2) \in \boldsymbol{\mathcal{SI}}$. Applying directly the theory of the previous section to $S(\lambda, t_2)$ leads to a new function which need not belong to $\boldsymbol{\mathcal{SI}}$. Instead, we apply the Schur algorithm to the $\sigma_1(t_2^0)$-inner function $S(\lambda, t_2^0)$ for some preassigned $t_2^0 \in \mathrm{I}$, and obtain a simpler (in terms of McMillan degree) $\sigma_1(t_2^0)$-inner function $S_0(\lambda, t_2^0)$. We use Theorem 4.1 to obtain an element in a class $\boldsymbol{\mathcal{SI}}$ from $S_0(\lambda, t_2^0)$. We call this procedure the tangential Schur algorithm for the class $\boldsymbol{\mathcal{SI}}$. We study in Section 5 the coefficients (called the *Markov moments*) $H_i(t_2)$ of the expansion of $S(\lambda, t_2)$ around $\lambda = \infty$

$$S(\lambda, t_2) = I_p - \sum_{i=0}^{\infty} \frac{1}{\lambda^{i+1}} H_i(t_2) \tag{1.2}$$

It turns out that the first Markov moment $H_0(t_2)$ satisfies the Lyapunov equation

(1.3)
$$\gamma_*(t_2) - \gamma(t_2) = \sigma_2(t_2) H_0(t_2) - \sigma_1(t_2) H_0(t_2) \sigma_1^{-1}(t_2) \sigma_2(t_2), \quad t_2 \in \mathrm{I},$$

which means that given the functions $\sigma_1, \sigma_2, \gamma$ and H_0 on I, one can uniquely reconstruct γ_*, and, as a result of Theorem 4.1, there will

exists a unique function $S(\lambda, t_2)$ ($t_2 \in \mathrm{I}$) with the given Markov parameters. We prove the following theorem 5.5 on the structure of Markov moments:
Theorem *Suppose that we are given moments $H_i(t_2)$, defined in a neighborhood of the point $t_2^0 \in \mathrm{I}$. Then there exists $n_0 \leq p^2$ such that each element H_{i+1} is uniquely determined from $H_0, \ldots, H_i$ using the algebraic formulas* (5.5)*, and n_0 LDEs with arbitrary initial conditions, obtained from* (5.4)*. Moreover, n_0 equations must be satisfied by the elements of $\gamma_*(t_2)$.*
Starting from the Markov coefficients at infinity, constructed under conditions of Theorem 5.5 we recover in Section 6 the transfer function $S(\lambda, t_2)$. In the arguments we make use of a Krein-space realization theorem of Dijksma–Langer–de Snoo for analytic functions at infinity [DLdeS, Theorem 3]. By a counterexample, we show that it is not always possible to reconstruct the function using Hilbert space realizations.
In Section 7 we study two generalizations of Nevanlinna Pick interpolation problem in the class $\boldsymbol{\mathcal{SI}}$. Let us recall that in the classical Nevanlinna Pick interpolation problem [N, P] Schur analysis plays a special role; see for instance [FK], [Dy], [A]. Schur analysis gives a parametrization of all solutions (when they exist) for the given data. The first problem 1.4 is similar to the non commutative and Riemann surface cases and involves specifying the exact class of input-output mappings, and a finite number of inputs that are to be mapped to the corresponding outputs. In our case this yields fixing vessels parameters and a set of values $w_i, i = 1, \ldots, N$ and corresponding solutions $\xi_i(t_2)$ and $\eta_i(t_2)$ of the input and the output LDEs, respectively:

Problem 1.4 (Nevanlinna-Pick interpolation)**.** *Let $\sigma_1, \sigma_2, \gamma, \gamma_*$ be vessel parameters and let I be an interval. Let $N \in \mathbb{N}$ and $w_i, i = 1, \ldots, N$ be complex numbers. Suppose also that N input functions $\xi_i(t_2)$ satisfying* (2.16) *with corresponding spectral parameters w_i's, and N output functions $\eta_i(t_2)$ satisfying* (2.17) *with the given w_i's.*

(1) *Give sufficient and necessary conditions, so that there exists $S(\lambda, t_2) \in \boldsymbol{\mathcal{SI}}$ such that $S(w_j, t_2)\xi_i(t_2) = \eta_i(t_2)$, $j = 1, \ldots, N$, on a sub-interval of I.*
(2) *Describe the set of all solutions for this problem.*

The second problem uses the fact that we can also specify the data for different values of t_2 and is more similar to the classical one:

Problem 1.5. *Given $\mathbb{C}^{p\times p}$-valued functions $\sigma_1, \sigma_2, \gamma$ defined on* I*, and given N quadruples $< t_2^j, w_j, \xi_j, \eta_j >$, where $t_2^j \in I, w_j \in \mathbb{C}_+$, $\xi_j, \eta_j \in \mathbb{C}^{1\times p}$ $j = 1, \dots N$, then:*

(1) *Give sufficient and necessary conditions, so that there exists γ_* and $S(\lambda, t_2) \in \boldsymbol{\mathcal{SI}}$ such that $S(w_j, t_2^j)\xi_j = \eta_j$, $j = 1, \dots, N$, on a sub-interval of* I *containing all the t_2^j.*

(2) *Describe the set of all solutions for this problem.*

If all the values $t_2^j = t_2^0$ are equal, we have to find a function $S(\lambda, t_2^0)$ satisfying $S(w_j, t_2^0)\xi_j = \eta_j$. Thus, the above problem is a generalization of the classical Nevanlinna-Pick interpolation problem. We also remark that we do not address the question of describing the set of all solutions.
Remarks: Some of the results presented here have been announced in [AMV].

2. t_1 INVARIANT CONSERVATIVE $2D$ SYSTEMS.

The material in this section is taken from [M, MV1, MVc], where proofs and more details can be found. The origin of this theory can be found in the paper [Li].

2.1. **Definition.** An overdetermined conservative t_1-invariant $2D$ system is a linear input-state-output (i/s/o) system, which consists of operators depending only on the variable t_2 and is of the following form:

$$I\Sigma : \begin{cases} \frac{\partial}{\partial t_1} x(t_1, t_2) = A_1(t_2)x(t_1, t_2) + \widetilde{B}_1(t_2)u(t_1, t_2) \\ x(t_1, t_2) = F(t_2, t_2^0)x(t_1, t_2^0) + \int\limits_{t_2^0}^{t_2} F(t_2, s)\widetilde{B}_2(s)u(t_1, s)ds \\ y(t_1, t_2) = u(t_1, t_2) - \widetilde{B}(t_2)^* x(t_1, t_2), \end{cases} \tag{2.1}$$

where the variable t_1 belongs to $\mathbb{R}$, and the variable t_2 belongs to some interval I. Furthermore, the input $u(t_1, t_2)$ and the output $y(t_1, t_2)$ take values in some Hilbert space $\mathcal{E}$ and the state $x(t_1, t_2)$ takes values in the Hilbert space $\mathcal{H}_{t_2}$. We assume that $u(t_1, t_2)$ and $y(t_1, t_2)$ are continuous functions of each variable when the other variable is fixed. The operators of the system are supposed to satisfy the following:

Assumptions 2.1.

(1) *$A_1(t_2) : \mathcal{H}_{t_2} \to \mathcal{H}_{t_2}$, and $\widetilde{B}(t_2) : \mathcal{E} \to \mathcal{H}_{t_2}$ are bounded operators for all t_2,*

(2) *The functions $\sigma_1, \sigma_2, \gamma, \gamma_* : \mathcal{E} \to \mathcal{E}$, are continuous in the operator norm topology.*

(3) $\sigma_1(t_2)$ *is an invertible operator for every* $t_2 \in \mathrm{I}$.
(4) $F(t,s)$ *is an evolution continuous semi-group.*

For continuous inputs $u(t_1,t_2)$, the inner state is continuously differentiable. Requiring now the invariance of the system transition from (t_1^0,t_2^0) to (t_1,t_2) via the points (t_1^0,t_2) and (t_1,t_2^0) respectively, is equivalent to the equality of second order partial derivatives of $x(t_1,t_2)$:

$$\frac{\partial^2}{\partial t_1 \partial t_2} x(t_1,t_2) = \frac{\partial^2}{\partial t_2 \partial t_1} x(t_1,t_2). \tag{2.2}$$

Substituting in this equality the system equations we obtain that for the free evolution $u(t_1,t_2)=0$ the so called Lax equation holds

$$A_1(t_2) = F(t_2,t_2^0) A_1(t_2^0) F(t_2^0,t_2). \tag{2.3}$$

Inserting (2.3) into (2.2) we see that the input $u(t_1,t_2)$ has to satisfy the following PDE

$$\begin{aligned}&\widetilde{B}(t_2)\sigma_2(t_2)\tfrac{\partial}{\partial t_1}u(t_1,t_2) - \widetilde{B}(t_2)\sigma_1(t_2)\tfrac{\partial}{\partial t_2}u(t_1,t_2) - \\ &\big(A_1(t_2)\widetilde{B}(t_2)\sigma_2(t_2) + F(t_2,t_2^0)\tfrac{\partial}{\partial t_2}[F(t_2^0,t_2)\widetilde{B}(t_2)\sigma_1(t_2)]\big)u(t_1,t_2) = 0.\end{aligned}$$

Assuming the existence of a function $\gamma(t_2)$ satisfying

(2.4)
$$A_1(t_2)\widetilde{B}(t_2)\sigma_2(t_2) + F(t_2,t_2^0)\frac{\partial}{\partial s}[F(t_2^0,t_2)\widetilde{B}(t_2)\sigma_1(t_2)] = -\widetilde{B}(t_2)\gamma(t_2)$$

we obtain that it is enough that $u(t_1,t_2)$ satisfies the PDE

$$\sigma_2(t_2)\frac{\partial}{\partial t_1}u(t_1,t_2) - \sigma_1(t_2)\frac{\partial}{\partial t_2}u(t_1,t_2) + \gamma(t_2)u(t_1,t_2) = 0. \tag{2.5}$$

The output $y(t_1,t_2)$ should satisfy the *output compatibility condition* of the same type as for the input compatibility condition (2.5), namely:

$$\sigma_2(t_2)\frac{\partial}{\partial t_1}y(t_1,t_2) - \sigma_1(t_2)\frac{\partial}{\partial t_2}y(t_1,t_2) + \gamma_*(t_2)y(t_1,t_2) = 0. \tag{2.6}$$

Inserting here $y(t_1,t_2) = u(t_1,t_2) - \widetilde{B}(t_2)^* x(t_2,t_2)$ we obtain that

$$\begin{aligned}(2.7)\qquad 0 &= \sigma_2(t_2)\widetilde{B}(t_2)^* A_1(t_2) F(t_2,t_2^0) - \\ &\quad -\sigma_1(t_2)\frac{\partial}{\partial t_2}[\widetilde{B}(t_2)^* F(t_2,t_2^0)] + \gamma_*(t_2)\widetilde{B}(t_2)^* F(t_2,t_2^0) \\ (2.8)\ \gamma(t_2) &= \sigma_1(t_2)\widetilde{B}(t_2)^*\widetilde{B}(t_2)\sigma_2(t_2) - \\ &\quad -\sigma_2(t_2)\widetilde{B}(t_2)^*\widetilde{B}(t_2)\sigma_1(t_2) + \gamma_*(t_2).\end{aligned}$$

The fact that the system is lossless comes from the requirement of the so called *energy balance* equations:

$$\frac{\partial}{\partial t_i}\langle x(t_1,t_2),x(t_1,t_2)\rangle_{\mathcal{H}_{t_2}}+\langle\sigma_i(t_2)y(t_1,t_2),y(t_1,t_2)\rangle_{\mathcal{E}}=$$
$$=\langle\sigma_i(t_2)u(t_1,t_2),u(t_1,t_2)\rangle_{\mathcal{E}},\ i=1,2,$$

which means that the energy of the output is distributed between the energy of the input and the change of the energy of the state of the system. Immediate consequences of this requirement are

$$0 = A_1(t_2)+A_1^*(t_2)+\widetilde{B}(t_2)\sigma_1(t_2)\widetilde{B}(t_2)^*, \tag{2.9}$$
$$\frac{d}{dt_2}[F^*(t_2,t_2^0)F(t_2,t_2^0)] = F^*(t_2,t_2^0)\widetilde{B}(t_2)^*\sigma_2(t_2)\widetilde{B}(t_2)F(t_2,t_2^0). \tag{2.10}$$

In this manner we obtain the notion of *conservative vessel in the integral form*, which is a collection of operators and spaces

$$\mathfrak{V}=(A_1(t_2),F(t_2,t_2^0),\widetilde{B}(t_2);\sigma_1(t_2),\sigma_2(t_2),\gamma(t_2),\gamma_*(t_2);\mathcal{H}_{t_2},\mathcal{E})$$

where the operators satisfy the regularity assumptions 2.1, and the following vessel conditions:

$$0=A_1(t_2)+A_1^*(t_2)+\widetilde{B}(t_2)^*\sigma_1(t_2)\widetilde{B}(t_2) \tag{2.9}$$
$$\begin{aligned}&\|F(t_2,t_2^0)x(t_1,t_2^0)\|^2-\|x(t_1,t_2^0)\|^2=\\&\qquad=\textstyle\int_{t_2^0}^{t_2}\langle\sigma_2(s)\widetilde{B}(s)x(t_1,s),\widetilde{B}(s)x(t_1,s)\rangle ds\end{aligned} \tag{2.10}$$
$$F(t_2,t_2^0)A_1(t_2^0)=A_1(t_2)F(t_2,t_2^0) \tag{2.3}$$
$$\begin{aligned}0=&\tfrac{d}{dt_2}(F(t_2^0,t_2)\widetilde{B}(t_2)\sigma_1(t_2))+\\&+F(t_2^0,t_2)A_1(t_2)\widetilde{B}(t_2)\sigma_2(t_2)+F(t_2^0,t_2)\widetilde{B}(t_2)\gamma(t_2)\end{aligned} \tag{2.4}$$
$$\begin{aligned}0=&\sigma_1(t_2)\tfrac{\partial}{\partial t_2}[\widetilde{B}(t_2)^*F(t_2,t_2^0)]-\\&-\sigma_2(t_2)\widetilde{B}(t_2)^*A_1(t_2)F(t_2,t_2^0)-\gamma_*(t_2)\widetilde{B}(t_2)^*F(t_2,t_2^0)\end{aligned} \tag{2.7}$$
$$\begin{aligned}\gamma(t_2)=&-\sigma_2(t_2)\widetilde{B}(t_2)^*\widetilde{B}(t_2)\sigma_1(t_2)+\\&+\sigma_1(t_2)\widetilde{B}(t_2)^*\widetilde{B}(t_2)\sigma_2(t_2)+\gamma_*(t_2)\end{aligned} \tag{2.8}$$

In order to simplify some notations we introduce the following definition.

Definition 2.2. *Let*

$$\mathbf{U}=\mathbf{U}(\lambda_1,\lambda_2;t_2)=\sigma_2(t_2)\lambda_1-\sigma_1(t_2)\lambda_2+\gamma(t_2),$$

and similarly

$$\mathbf{U}_*=\mathbf{U}_*(\lambda_1,\lambda_2;t_2)=\sigma_2(t_2)\lambda_1-\sigma_1(t_2)\lambda_2+\gamma_*(t_2).$$

The vessel $\mathfrak{V}$ is naturally associated to the system (2.1)

$$\Sigma : \begin{cases} \frac{\partial}{\partial t_1} x(t_1,t_2) = A_1(t_2)\ x(t_1,t_2) + \widetilde{B}(t_2)\sigma_1(t_2)\ u(t_1,t_2) \\ x(t_1,t_2) = F(t_2,t_2^0)x(t_1,t_2^0) + \int\limits_{t_2^0}^{t_2} F(t_2,s)\widetilde{B}(s)\sigma_2(s)u(t_1,s)ds \\ y(t_1,t_2) = u(t_1,t_2) - \widetilde{B}(t_2)^*\ x(t_1,t_2). \end{cases}$$

with inputs and outputs satisfying the compatibility conditions (2.5) and (2.6), i.e. satisfy:

$$\mathbf{U}(\frac{\partial}{\partial t_1}, \frac{\partial}{\partial t_2}; t_2)u(t_1,t_2) = 0, \quad \mathbf{U}_*(\frac{\partial}{\partial t_1}, \frac{\partial}{\partial t_2}; t_2)y(t_1,t_2) = 0$$

The theory of such vessels, developed in [M, MV1] enables to find a more convenient form of the vessel. Denoting $\mathcal{H} = \mathcal{H}_{t_2^0}$, $A_1 = A_1(t_2^0)$, $F^*(t_2,t_2^0)F(t_2,t_2^0) = \mathbb{X}^{-1}(t_2)$ and $B(t_2) = F(t_2^0,t_2)\widetilde{B}(t_2)$, we shall obtain the following notion, first introduced in [M2].

Definition 2.3. *A (differential) conservative vessel associated to the vessel parameters is a collection of operators and spaces*

$$\mathfrak{V} = (A_1, B(t_2), \mathbb{X}(t_2); \sigma_1(t_2), \sigma_2(t_2), \gamma(t_2), \gamma_*(t_2); \mathcal{H}, \mathcal{E}), \tag{2.11}$$

where the operators satisfy the following vessel conditions:

$$0 = \frac{d}{dt_2}(B(t_2)\sigma_1(t_2)) + A_1 B(t_2)\sigma_2(t_2) + B(t_2)\gamma(t_2), \tag{2.12}$$

$$A_1\mathbb{X}(t_2) + \mathbb{X}(t_2)A_1^* = B(t_2)\sigma_1(t_2)B(t_2)^*, \tag{2.13}$$

$$\frac{d}{dt_2}\mathbb{X}(t_2) = B(t_2)\sigma_2(t_2)B(t_2)^*, \tag{2.14}$$

$$\begin{aligned} \gamma_*(t_2) = \gamma(t_2) + \sigma_2(t_2)B(t_2)^*\mathbb{X}^{-1}(t_2)B(t_2)\sigma_1(t_2) - \\ -\sigma_1(t_2)B(t_2)^*\mathbb{X}^{-1}(t_2)B(t_2)\sigma_2(t_2) \end{aligned} \tag{2.15}$$

It turns out that Lyapunov equation (2.13) is partially redundant.

Lemma 2.4. *Suppose that $B(t_2)$ satisfies* (2.12) *and $\mathbb{X}(t_2)$ satisfies* (2.14)*, then if the Lyapunov equation* (2.13)

$$A_1\mathbb{X}(t_2) + \mathbb{X}(t_2)A_1^* + B(t_2)\sigma_1(t_2)B(t_2)^* = 0$$

holds for a fixed t_2^0, then it holds for all t_2. If $\mathbb{X}(t_2^0) = \mathbb{X}^*(t_2^0)$, then $\mathbb{X}(t_2) = \mathbb{X}(t_2)^*$ for all t_2.

Proof: By differentiating the left hand side of (2.13), we will obtain that it is zero. The derivative of $\mathbb{X}(t_2)$ is selfadjoint. □

This representation of a vessel is the most convenient when one focuses on the notion of transfer function, as we do in the next subsection.

2.2. **Transfer function.** Performing a separation of variables as follows

$$\begin{aligned} u(t_1,t_2) &= u_\lambda(t_2)e^{\lambda t_1}, \\ x(t_1,t_2) &= x_\lambda(t_2)e^{\lambda t_1}, \\ y(t_1,t_2) &= y_\lambda(t_2)e^{\lambda t_1}, \end{aligned}$$

we arrive at the notion of a transfer function. Note that $u(t_1,t_2)$ and $y(t_1,t_2)$ satisfy PDEs, but $u_\lambda(t_2)$ and $y_\lambda(t_2)$ are solutions of LDEs with spectral parameter λ,

$$\mathbf{U}(\lambda, \frac{\partial}{\partial t_2}; t_2)u(t_1,t_2) = 0, \tag{2.16}$$

$$\mathbf{U}(\lambda, \frac{\partial}{\partial t_2}; t_2)y(t_1,t_2) = 0. \tag{2.17}$$

The corresponding i/s/o system becomes

$$\begin{cases} x_\lambda(t_2) = (\lambda I - A_1(t_2))^{-1}B(t_2)\sigma_1(t_2)u_\lambda(t_2) \\ \frac{\partial}{\partial t_2}x_\lambda(t_2) = F(t_2,t_2^0)x_\lambda(t_2^0) + \int\limits_{t_2^0}^{t_2} F(t_2,s)\widetilde{B}_2(s)u_\lambda(s)ds \\ y_\lambda(t_2) = u_\lambda(t_2) - \widetilde{B}(t_2)^*x_\lambda(t_2) \end{cases}$$

The output $y_\lambda(t_2) = u_\lambda(t_2) - \widetilde{B}(t_2)^*x_\lambda(t_2)$ may be found from the first i/s/o equation:

$$y_\lambda(t_2) = S(\lambda,t_2)u_\lambda(t_2),$$

using the *transfer function*

$$\begin{aligned} S(\lambda,t_2) &= I - \widetilde{B}(t_2)^*(\lambda I - A_1(t_2))^{-1}\widetilde{B}(t_2)\sigma_1(t_2) \\ &= I - \widetilde{B}(t_2)^*(\lambda I - F(t_2,t_2^0)A_1(t_2^0)F(t_2^0,t_2))^{-1}\widetilde{B}(t_2)\sigma_1(t_2) \\ &= I - \widetilde{B}(t_2)^*F(t_2,t_2^0)(\lambda I - A_1(t_2^0))^{-1}F(t_2^0,t_2)\widetilde{B}(t_2)\sigma_1(t_2) \\ &= I - \widetilde{B}(t_2)^*F^*(t_2^0,t_2)F^*(t_2,t_2^0)F(t_2,t_2^0)(\lambda I - A_1)^{-1}B(t_2)\sigma_1(t_2) \\ &= I - B(t_2)^*\mathbb{X}^{-1}(t_2)(\lambda I - A_1)^{-1}B(t_2)\sigma_1(t_2) \end{aligned}$$

and we obtain that

$$S(\lambda,t_2) = I - B(t_2)^*\mathbb{X}^{-1}(t_2)(\lambda I - A_1)^{-1}B(t_2)\sigma_1(t_2). \tag{2.18}$$

Proposition 2.5. *The transfer function $S(\lambda,t_2)$ defined by* (2.18) *has the following properties:*

(1) *For all t_2, $S(\lambda,t_2)$ is an analytic function of λ in the neighborhood of ∞, where it satisfies:*

$$S(\infty,t_2) = I_p$$

(2) *For all λ, $S(\lambda,t_2)$ is a continuous function of t_2.*

(3) *For λ in the domain of analyticity of $S(\lambda, t_2)$:*

$$S(\lambda, t_2)^* \sigma_1(t_2) S(\lambda, t_2) \leq \sigma_1(t_2), \quad \Re\lambda > 0, \tag{2.19}$$

and

$$S(\lambda, t_2)^* \sigma_1(t_2) S(\lambda, t_2) = \sigma_1(t_2), \quad \Re\lambda = 0. \tag{2.20}$$

(4) *Maps solutions of the input LDE* (2.16) *with spectral parameter λ to the output LDE* (2.17) *with the same spectral parameter.*

Proof: These properties are easily checked, and follow from the definition of $S(\lambda, t_2)$:

$$S(\lambda, t_2) = I - \widetilde{B}(t_2)^* (\lambda I - A_1(t_2))^{-1} \widetilde{B}(t_2) \sigma_1(t_2).$$

The function $S(\lambda, t_2)$ is analytic for $\lambda > \|A_1(t_2)\|$ and since all the operators are bounded, we have $S(\infty, t_2) = I_p$. The second property follows from the regularity assumptions 2.1. The third property follows from straightforward calculations:

$$\begin{aligned}&S(\lambda, t_2)^* \sigma_1(t_2) S(\lambda, t_2) - \sigma_1(t_2) = \\ &-2\Re(\lambda) \sigma_1(t_2) \widetilde{B}(t_2)^* (\bar{\lambda} I - A_1^*(t_2))^{-1} (\lambda I - A_1(t_2))^{-1} \widetilde{B}(t_2) \sigma_1(t_2)\end{aligned}$$

Here the sign of $-\Re(\lambda)$ determines the sign of $S(\lambda, t_2)^* \sigma_1(t_2) S(\lambda, t_2) - \sigma_1(t_2)$ and thus the third property is obtained. The fourth property follows directly from our construction. □

Remark: When $\dim \mathcal{H} < \infty$, we obtain that $S(\lambda, t_2)$ is a rational function of λ for every t_2.

It is an interesting fact that also the converse of Proposition 2.5 holds; see [M], [MVc, chapter 5].

Theorem 2.6. *For any function of two variables $S(\lambda, t_2)$, satisfying the conditions of Proposition 2.5, there is a conservative t_1-invariant vessel whose transfer function is $S(\lambda, t_2)$.*

We define the class of transfer functions mentioned in the introduction as follows:

Definition 2.7 ([MVc])**.** *The class $\boldsymbol{\mathcal{SI}} = \boldsymbol{\mathcal{SI}}(\mathbf{U}(\lambda, \frac{\partial}{\partial t_2}; t_2), \mathbf{U}_*(\lambda, \frac{\partial}{\partial t_2}; t_2))$ consists of functions $S(\lambda, t_2)$ of two variables, which*

(1) *are analytic in a neighborhood of $\lambda = \infty$ for all t_2 and where it holds $S(\infty, t_2) = I_p$,*
(2) *are continuous as functions of t_2 for all λ,*
(3) *satisfy* (2.19) *and* (2.20) *in the domain of analyticity of S,*

(4) *map solutions of the input LDE (2.16) with spectral parameter λ to the output LDE (2.17) with the same spectral parameter*

Recall (see [CoLe]) that to every LDE can be associated an invertible matrix (or operator) function $\Phi(t_2, t_2^0)$, called the *fundamental solution*, which takes value I at some preassigned value t_2^0 and such that any other solution $u(t_2)$ of the LDE, with initial condition $u(t_2^0) = u_0$ is of the form

$$u(t_2) = \Phi(t_2, t_2^0)u_0.$$

Let $\Phi(\lambda, t_2, t_2^0)$ and $\Phi_*(\lambda, t_2, t_2^0)$ be the fundamental solutions of the input LDE (2.16) and the output LDE (2.17) respectively, where we have added in the notation the dependence in λ. Then,

$$S(\lambda, t_2)\Phi(\lambda, t_2, t_2^0) = \Phi_*(\lambda, t_2, t_2^0)S(\lambda, t_2^0) \tag{2.21}$$

and consequently $S(\lambda, t_2)$ satisfies the following LDE

$$\begin{aligned} \frac{\partial}{\partial t_2}S(\lambda, t_2) = \sigma_1^{-1}(t_2)(\sigma_2(t_2)\lambda + \gamma_*(t_2))S(\lambda, t_2)- \\ -S(\lambda, t_2)\sigma_1^{-1}(t_2)(\sigma_2(t_2)\lambda + \gamma(t_2)). \end{aligned} \tag{2.22}$$

The symmetry conditions (2.19) and (2.20), which are a result of Lyapunov equation (3.8) gives us

$$S(\lambda, t_2) = \sigma_1^{-1}(t_2)S^{-1*}(-\bar{\lambda}, t_2)\sigma_1(t_2) \tag{2.23}$$

3. Schur analysis for the classical case

One of the approaches to the tangential Schur algorithm for a rational matrix function $S(\lambda) \in \mathbb{C}^{p\times p}$ is based on the theory of reproducing kernel Hilbert spaces of the kind introduced by de Branges and Rovnyak; see [dBR1], [dBR2], [Dy] for information on these spaces. The paper [AD] considers the case of column-valued functions. In this section we adapt the results of [AD] to the case of row-valued functions. We also present a new formula for a realization of a Schur function after implementation of the tangential Schur algorithm.

3.1. **Schur functions and Reproducing Kernel Hilbert spaces.**

In this section σ_1 denotes a fixed self-adjoint and invertible (but not necessarily unitary) matrix in $\mathbb{C}^{p\times p}$. Let $S(\lambda)$ be a rational function, σ_1-inner in the open right half plane, i.e.

$$S(\lambda)^*\sigma_1 S(\lambda) - \sigma_1 \leq 0$$

at all points in the domain of analyticity $\Omega(S)$ of S in $\mathbb{C}_+$, and

$$S(\lambda)^*\sigma_1 S(\lambda) - \sigma_1 = 0$$

at all points on the imaginary axis where S is defined. Then, the kernel

$$K_S(\lambda, w) = \frac{\sigma_1 - S(w)^*\sigma_1 S(\lambda)}{\bar{w} + \lambda} \tag{3.1}$$

is positive for $\lambda, w \in \Omega(S)$, and the space of rational $\mathbb{C}^{1\times p}$-valued functions

$$\mathcal{H}(S) = \{\sum_{i=1}^{n} \alpha_i c_i K_S(\lambda, w_i) \mid \alpha_1 \in \mathbb{C}, w_i \in \Omega, c_i \in \mathbb{C}^{1\times p}\}$$

is finite dimensional. These well-known facts can be proved using realization theory; see for instance [AG]. Furthermore, $\mathcal{H}(S)$ is the reproducing kernel Hilbert space, associated to the kernel $K_S(\lambda, w)$. The inner product is defined by

$$\langle cK_S(\lambda, \nu), dK_S(\lambda, w)\rangle_{\mathcal{H}_S} = \langle cK_S(w, \nu)\rangle_{C^{1\times p}} = cK_S(w, \nu)d^*.$$

For an arbitrary $f(\lambda) \in \mathcal{H}(S)$ we have the reproducing kernel property

$$\langle f(\lambda), \xi K(\lambda, w)\rangle_{\mathcal{H}_S} = f(w)\xi^*.$$

More generally, let now $\mathcal{M}$ be a finite dimensional Hilbert space of $\mathbb{C}^{1\times p}$-valued functions defined in some set Ω, and let $\{f_1(\lambda), \ldots, f_N(\lambda)\}$ be a basis of $\mathcal{M}$. Let $\mathbb{X} \in \mathbb{C}^{p\times p}$ denote the Gram matrix with ℓ, j entry given by

$$\mathbb{X}_{\ell,j} = \langle f_j(\lambda), f_\ell(\lambda)\rangle_{\mathcal{M}}, \quad \ell, j = 1, \ldots, p. \tag{3.2}$$

It is easily seen that the space $\mathcal{M}$ is a reproducing kernel Hilbert space with kernel given by the formula

$$K(\lambda, w) = \begin{bmatrix} f_1(w)^* & \cdots & f_N(w)^* \end{bmatrix} \mathbb{X}^{-1} \begin{bmatrix} f_1(\lambda) \\ \vdots \\ f_N(\lambda) \end{bmatrix}. \tag{3.3}$$

We set

$$F(\lambda) = \begin{bmatrix} f_1(\lambda) \\ \vdots \\ f_p(\lambda) \end{bmatrix}. \tag{3.4}$$

Assume now that $\mathcal{M}$ consists of rational functions, defined on a set $\Omega(\mathcal{M})$. For $\alpha \in \Omega(\mathcal{M})$ the backward-shift operator R_α is defined by

$$R_\alpha(f(\lambda)) = \frac{f(\lambda) - f(\alpha)}{\lambda - \alpha}.$$

Suppose that:

(1) The space $\mathcal{M}$ is invariant under the action of R_α,

(2) The functions $f_i(\lambda)$ has the property that $f_i(\infty) = 0$, i.e., $F(\infty) = 0$,

then the function F given by (3.4) can be written as

$$F(\lambda) = (\lambda I - A)^{-1} B\sigma_1,$$

for suitably chosen matrices A, B. In this special case, formula (3.3) takes the form

$$K(\lambda, w) = \sigma_1 B^*(\bar{w}I - A^*)^{-1}\mathbb{X}^{-1}(\lambda I - A)^{-1}B\sigma_1. \tag{3.5}$$

As mentioned at the beginning of this section we are interested in kernels of the form (3.1) for some σ_1-inner rational function S. We now recall the characterization of these spaces, and first note the following: equation (3.1) leads to

$$\frac{\sigma_1 - S(w)^*\sigma_1 S(\lambda)}{\bar{w} + \lambda} = F(w)^*\mathbb{X}^{-1}F(\lambda). \tag{3.6}$$

If S is analytic at infinity and satisfies there $S(\infty) = I_p$, and letting $w \to \infty$ in this equation, we obtain the formula

$$S(\lambda) = I_p - B^*\mathbb{X}^{-1}(\lambda I - A)^{-1}B\sigma_1. \tag{3.7}$$

Theorem 3.1. *Let $\mathcal{M}$ be a finite dimensional Hilbert space of $\mathbb{C}^{1\times p}$-valued rational functions, which are zero at infinity. Suppose that $R_\alpha\mathcal{M} \subset \mathcal{M}$ for $\alpha \in \Omega(\mathcal{M})$ and let $\mathbb{X}$ be its Gram matrix with respect to $F(\lambda)$. Then $\mathcal{M} = \mathcal{H}(S)$ for S defined by* (3.7) *if and only if the Lyapunov equation*

$$A\mathbb{X} + \mathbb{X}A^* + B\sigma_1 B^* = 0 \tag{3.8}$$

holds.

When the spectrum of the operator A is in the open left half plane $\Re\lambda < 0$, one has

$$\mathcal{H}(S) = \mathbf{H}_{2,\sigma_1} \ominus \mathbf{H}_{2,\sigma_1} S,$$

where $\mathbf{H}_{2,\sigma_1}$ is the Hardy space $\mathbf{H}_2^p$ with the inner product

$$[f, g]_{\mathbf{H}_{2,\sigma_1}} = \langle f, g\sigma_1^{-1}\rangle_{\mathbf{H}_2^p}.$$

We set $J = \begin{bmatrix} -\sigma_1 & 0 \\ 0 & \sigma_1 \end{bmatrix}$. Note that J is both invertible and self-adjoint, and one can define J-inner rational functions. Let $\Theta = \begin{bmatrix} \Theta_{11} & \Theta_{12} \\ \Theta_{21} & \Theta_{22} \end{bmatrix}$ be a J-inner rational function; we introduce the linear fractional transformation

$$T_\Theta(W) = (\Theta_{11} + W\Theta_{21})^{-1}(\Theta_{12} + W\Theta_{22}).$$

Theorem 3.2. *Let S and Θ be respectively σ_1-inner J-inner rational functions. Then there exists a σ_1-inner rational function W such that $S = T_\Theta(W)$ if and only if the map*

$$F \mapsto F \begin{bmatrix} -S(\lambda) \\ I_p \end{bmatrix} \tag{3.9}$$

is a contraction from $\mathcal{H}(\Theta)$ to $\mathcal{H}(S)$.

This theorem originates with the work of de Branges and Rovnyak (see [dBR1, Theorem 13 p. 305]), where it is proved in a more general setting, and is one of the key ingredients to the reproducing kernel approach to the Schur algorithm. The proof is the same as for column-valued functions, and is omitted.

As a special case of the previous theorem we have:

Corollary 3.3. *Given $S \in \boldsymbol{\mathcal{RS}}$, $w_1, \ldots, w_n \in \mathbb{C}_+$, and row vectors $\xi_1, \ldots, \xi_n \in \mathbb{C}^{1\times p}$, define*

$$B = \begin{bmatrix} -\xi_1 S(w_1)^* & \xi_1 \\ \vdots & \vdots \\ -\xi_n S(w_n)^* & \xi_n \end{bmatrix}, \quad A_1 = \operatorname{diag}[-w_1^*, \ldots, -w_n^*].$$

Let $\mathcal{M}$ be a Hilbert space of row vectors spanned by the rows of the matrix-valued function

$$F(\lambda) = (\lambda I - A_1)^{-1} BJ.$$

Let $\mathbb{X}$ be the solution of the Lyapunov equation

$$A\mathbb{X} + \mathbb{X}A^* + BJB^* = 0 \tag{3.10}$$

and assume $\mathbb{X} > 0$. Let Θ be defined by

$$\Theta(\lambda) = I_{2p} - B^*\mathbb{X}^{-1}(\lambda I_n - A_1)^{-1} BJ. \tag{3.11}$$

Then there exists $S_0 \in \boldsymbol{\mathcal{RS}}$ such that $S = T_\Theta(S_0)$.

Proof: Let us denote by t the map (3.9). For an arbitrary element $f(\lambda) = \eta F(\lambda)$, we shall obtain that

$$\begin{aligned} tf(\lambda) &= \eta(\lambda I - A_1)^{-1} BJ \begin{bmatrix} -S(\lambda) \\ I_p \end{bmatrix} = \\ &= \eta \operatorname{diag}[\tfrac{1}{\lambda + \bar{w}_i}] \begin{bmatrix} -\xi_1 S(w_1)^* & \xi_1 \\ \vdots & \vdots \\ -\xi_n S(w_n)^* & \xi_n \end{bmatrix} \begin{bmatrix} \sigma_1 S \\ \sigma_1 \end{bmatrix} = \\ &= \textstyle\sum_i \frac{\eta_i}{\lambda + \bar{w}_i} \xi_i (\sigma_1 - S(w_i)^* \sigma_1 S(\lambda)) = \\ &= \textstyle\sum_i \xi_i \eta_i K_S(\lambda, w_i) \end{aligned}$$

and consequently,

$$\begin{aligned}\langle tf, tf\rangle &= \langle \sum_i \xi_i\eta_i K_S(\lambda, w_i), \sum_j \xi_j\eta_i K_S(\lambda, w_j)\rangle = \\ &= \sum_{ij} \eta_i\xi_i K_S(w_j, w_i)\eta_j^*\xi_j^* = \eta\mathbb{X}\eta^* = \\ &= \langle f, f\rangle.\end{aligned}$$

Thus t is an isometry and Theorem 3.2 allows to conclude. □

The following result shows that the assumption $\mathbb{X} > 0$ in the statement of Corollary 3.3 can always be achieved for $n = 1$.

Lemma 3.4. *Given $S \in \boldsymbol{\mathcal{RS}}$ which is not the function identically equal to I_p. Then there exist a pair $(\xi, w) \in \mathbb{C}^{1\times p} \times \mathbb{C}_+$ such that the corresponding $\mathbb{X} > 0$.*

Proof: We proceed by contradiction. Assume that for each $w \in \mathbb{C}_+$ and for each vector ξ, it holds that

$$\xi\sigma_1\xi^* = \xi S(w)^*\sigma_1 S(w)\xi^*.$$

or, equivalently,

$$\xi K_S(w, w)\xi^* = 0.$$

Then, $\xi K_S(\lambda, w) \equiv 0$, and for each $f \in \mathcal{H}(S)$

$$\xi f(w) =< f(\lambda, \xi K_S(\lambda, w) >= 0.$$

The space $\mathcal{H}(S)$ is thus trivial, and its kernel is zero:

$$\frac{\sigma_1 - S(w)^*\sigma_1 S(\lambda)}{\lambda + \bar{w}} = 0,$$

from where we conclude that for each w, λ

$$S(\lambda) = \sigma_1^{-1} S^{-*}(w)\sigma_1.$$

and consequently, $S(\lambda) \equiv I_p$. □

3.2. **Analysis of the tangential Schur algorithm.** For $n = 1$, the matrix function in (3.11) becomes

(3.12)

$$\begin{aligned}\Theta(\lambda) &= I_{2p} - B^*\mathbb{X}^{-1}(\lambda I - A)^{-1}BJ = \\ &= I_{2p} - \begin{bmatrix} -\eta^* \\ \xi^* \end{bmatrix} \begin{bmatrix} -\eta & \xi \end{bmatrix} \begin{bmatrix} -\sigma_1 & 0 \\ 0 & \sigma_1 \end{bmatrix} \frac{1}{\mathbb{X}(\lambda + w^*)} = \\ &= \begin{bmatrix} I_p + \dfrac{\eta^*\eta\sigma_1}{\mathbb{X}(\lambda + w^*)} & \dfrac{\eta^*\xi\sigma_1}{\mathbb{X}(\lambda + w^*)} \\ -\dfrac{\xi^*\eta\sigma_1}{\mathbb{X}(\lambda + w^*)} & I_p - \dfrac{\xi^*\xi\sigma_1}{\mathbb{X}(\lambda + w^*)} \end{bmatrix} = \begin{bmatrix} \Theta_{11} & \Theta_{12} \\ \Theta_{21} & \Theta_{22} \end{bmatrix},\end{aligned}$$

where using the Lyapunov equation (3.10)

$$\begin{aligned} A\mathbb{X}+\mathbb{X}A^* &= \mathbb{X}(-w^*-w) = \\ &= -BJB^* = -\begin{bmatrix} -\eta & \xi \end{bmatrix}\begin{bmatrix} -\sigma_1 & 0 \\ 0 & \sigma_1 \end{bmatrix}\begin{bmatrix} -\eta^* \\ \xi^* \end{bmatrix} \end{aligned}$$

we find that

$$\mathbb{X} = \frac{\xi\sigma_1\xi^* - \eta\sigma_1\eta^*}{w+w^*}. \tag{3.13}$$

To the best of our knowledge, Theorem 3.5 below is new and uses a realization theorem of M. Livšic [BLi]. It will be of much use in the following sections.

Theorem 3.5. *Let $S_0 \in \boldsymbol{\mathcal{S}}$ with minimal realization*

$$S_0(\lambda) = I_p - B_0^*\mathbb{X}_0^{-1}(\lambda I - A_0)^{-1}B_0\sigma_1, \tag{3.14}$$

and let Θ be given by (3.12). Then $S = T_\Theta(S_0) \in \boldsymbol{\mathcal{RS}}$ and a minimal realization of S is given by

$$S(\lambda) = I_p - B_S^*\mathbb{X}_S^{-1}(\lambda I - A_S)^{-1}B_S\sigma_1,$$

where

$$B_S = \begin{bmatrix} B_0 \\ \eta - \xi \end{bmatrix} \tag{3.15}$$

$$\mathbb{X}_S = \begin{bmatrix} \mathbb{X}_0 & 0 \\ 0 & \mathbb{X} \end{bmatrix} = \begin{bmatrix} \mathbb{X}_0 & 0 \\ 0 & \dfrac{\xi\sigma_1\xi^* - \eta\sigma_1\eta^*}{w+w^*} \end{bmatrix} \tag{3.16}$$

$$A_S = \begin{bmatrix} A_0 & \dfrac{B_0\sigma_1\xi^*}{\mathbb{X}} \\ -\eta\sigma_1 B_0^*\mathbb{X}_0^{-1} & -w^* - \dfrac{\eta\sigma_1(\eta^*-\xi^*)}{\mathbb{X}} \end{bmatrix} \tag{3.17}$$

Proof: From the definition

$$\begin{aligned} S(\lambda) &= T_\Theta(S_0(\lambda)) = \\ &= (\Theta_{11} + S_0(\lambda)\Theta_{21})^{-1}(\Theta_{12} + S_0(\lambda)\Theta_{22}) = \\ &= (I_p + \frac{\eta^*\eta\sigma_1}{\mathbb{X}(\lambda+w^*)} - S_0(\lambda)\frac{\xi^*\eta\sigma_1}{\mathbb{X}(\lambda+w^*)})^{-1}\times \\ &\quad\times\left(\frac{\eta^*\xi\sigma_1}{\mathbb{X}(\lambda+w^*)} + S_0(\lambda)[I_p - \frac{\xi^*\xi\sigma_1}{\mathbb{X}(\lambda+w^*)}]\right). \end{aligned}$$

Let us denote here $S_0(\lambda)\xi^* = \alpha^*$; then the preceding expression becomes

$$\begin{aligned} S(\lambda) &= (I_p + \frac{\eta^*\eta\sigma_1}{\mathbb{X}(\lambda+w^*)} - \frac{\alpha^*\eta\sigma_1}{\mathbb{X}(\lambda+w^*)})^{-1}\times \\ &\quad \times(\frac{\eta^*\xi\sigma_1}{\mathbb{X}(\lambda+w^*)} + S_0(\lambda) - \frac{\alpha^*\xi\sigma_1}{\mathbb{X}(\lambda+w^*)}) = \\ &= (I_p + \frac{(\eta^*-\alpha^*)\eta\sigma_1}{\mathbb{X}(\lambda+w^*)})^{-1}(S_0(\lambda) + \frac{(\eta^*-\alpha^*)\xi\sigma_1}{\mathbb{X}(\lambda+w^*)}) = \end{aligned}$$

Using $(I+b^*a)^{-1} = I - b^*a\dfrac{1}{1+ab^*}$ we get

$$\begin{aligned} S(\lambda) &= (I_p - \frac{(\eta^*-\alpha^*)\eta\sigma_1}{\mathbb{X}(\lambda+w^*)}\frac{1}{1+\dfrac{\eta\sigma_1(\eta^*-\alpha^*)}{\mathbb{X}(\lambda+w^*)}})(S_0(\lambda) + \frac{(\eta^*-\alpha^*)\xi\sigma_1}{\mathbb{X}(\lambda+w^*)}) = \\ &= S_0(\lambda) - \frac{(\eta^*-S_0(\lambda)\xi^*)[\eta\sigma_1S_0(\lambda)-\xi\sigma_1]}{\mathbb{X}(\lambda+w^*)+\eta\sigma_1(\eta^*-S_0(\lambda)\xi^*)}. \end{aligned}$$

Suppose further that there is a realization of the form (3.14) for the function $S_0(\lambda)$. Inserting it here we shall obtain

$$\begin{aligned} S(\lambda) &= S_0(\lambda) - \frac{(\eta^*-S_0(\lambda)\xi^*)[\eta\sigma_1S_0(\lambda)-\xi\sigma_1]}{\mathbb{X}(\lambda+w^*)+\eta\sigma_1(\eta^*-S_0(\lambda)\xi^*)} = \\ &= I_p - B_0^*\mathbb{X}_0^{-1}(\lambda I - A_0)^{-1}B_0\sigma_1 - \\ &\quad - \frac{(\eta^*-S_0(\lambda)\xi^*)[\eta\sigma_1S_0(\lambda)-\xi\sigma_1]}{\mathbb{X}(\lambda+w^*)+\eta\sigma_1(\eta^*-S_0(\lambda)\xi^*)}. \end{aligned}$$

Let us denote

$$\begin{aligned} M &= \mathbb{X}(\lambda+w^*) + \eta\sigma_1(\eta^*-S_0(\lambda)\xi^*) = \\ &= \mathbb{X}(\lambda+w^*) + \eta\sigma_1(\eta^*-\xi^*) + B_0^*\mathbb{X}_0^{-1}(\lambda I - A_0)^{-1}B_0\sigma_1\xi^*. \end{aligned}$$

The preceding formula for $S(\lambda)$ becomes

$$\begin{aligned} S(\lambda) &= I_p - B_0^*\mathbb{X}_0^{-1}(\lambda I - A_0)^{-1}B_0\sigma_1 - \frac{(\eta^*-S_0(\lambda)\xi^*)[\eta\sigma_1S_0(\lambda)-\xi\sigma_1]}{\mathbb{X}(\lambda+w^*)+\eta\sigma_1(\eta^*-S_0(\lambda)\xi^*)} = \\ &= I_p - \frac{1}{M}\begin{bmatrix} B_0^* & \eta^*-\xi^* \end{bmatrix}\begin{bmatrix} \mathbb{X}_0^{-1} & 0 \\ 0 & 1 \end{bmatrix}\times \\ &\quad \times\begin{bmatrix} \alpha^{-1}M + \alpha^{-1}\beta\delta\alpha^{-1} & -\alpha^{-1}\beta \\ -\delta\alpha^{-1} & 1 \end{bmatrix}\begin{bmatrix} B_0 \\ \eta-\xi \end{bmatrix}\sigma_1, \end{aligned}$$

where

$$\begin{aligned} \alpha &= \lambda I - A_0 \\ \beta &= -B_0\sigma_1\xi^* \\ \delta &= \eta\sigma_1B_0^*\mathbb{X}_0^{-1}. \end{aligned}$$

By definition of M we have

$$M = \mathbb{X}(\lambda+w^*) + \eta\sigma_1(\eta^*-\xi^*) - \delta\alpha^{-1}\beta,$$

and hence we obtain from the formula

$$\begin{bmatrix} \alpha & \beta \\ \delta & D \end{bmatrix}^{-1} = \frac{1}{D-\delta\alpha^{-1}\beta} \begin{bmatrix} \alpha^{-1}(D-\delta\alpha^{-1}\beta)+\alpha^{-1}\beta\delta\alpha^{-1} & -\alpha^{-1}\beta \\ -\delta\alpha^{-1} & 1 \end{bmatrix}$$

that the last expression for $S(\lambda)$ is

$$\begin{aligned} S(\lambda) &= I_p - \frac{1}{M}\begin{bmatrix} B_0^* & \eta^*-\xi^* \end{bmatrix} \begin{bmatrix} \mathbb{X}_0^{-1} & 0 \\ 0 & 1 \end{bmatrix} \times \\ &\quad \times \begin{bmatrix} \alpha^{-1}M+\alpha^{-1}\beta\delta\alpha^{-1} & -\alpha^{-1}\beta \\ -\delta\alpha^{-1} & 1 \end{bmatrix} \begin{bmatrix} B_0 \\ \eta-\xi \end{bmatrix} \sigma_1 = \\ &= I_p - \begin{bmatrix} B_0^* & \eta^*-\xi^* \end{bmatrix} \begin{bmatrix} \mathbb{X}_0^{-1} & 0 \\ 0 & 1 \end{bmatrix} \times \\ &\quad \times \begin{bmatrix} \alpha & \beta \\ \delta & \mathbb{X}(\lambda+w^*)+\eta\sigma_1(\eta^*-\xi^*) \end{bmatrix}^{-1} \begin{bmatrix} B_0 \\ \eta-\xi \end{bmatrix} \sigma_1 = \\ &= I_p - \begin{bmatrix} B_0^* & \eta^*-\xi^* \end{bmatrix} \begin{bmatrix} \mathbb{X}_0^{-1} & 0 \\ 0 & \mathbb{X}^{-1} \end{bmatrix} \times \\ &\quad \times \begin{bmatrix} \lambda I - A_0 & -\dfrac{B_0\sigma_1\xi^*}{\mathbb{X}} \\ \eta\sigma_1 B_0^*\mathbb{X}_0^{-1} & \lambda+w^*+\dfrac{\eta\sigma_1(\eta^*-\xi^*)}{\mathbb{X}} \end{bmatrix}^{-1} \begin{bmatrix} B_0 \\ \eta-\xi \end{bmatrix} \sigma_1 . \end{aligned}$$

Thus we have obtained the realization (3.15)–(3.17). Furthermore, the Lyapunov equation (3.10) holds since

$$\begin{aligned} & A_S\mathbb{X}_S + \mathbb{X}_S A_S^* = \\ &= \begin{bmatrix} A_0 & \dfrac{B_0\sigma_1\xi^*}{\mathbb{X}} \\ -\eta\sigma_1 B_0^*\mathbb{X}_0^{-1} & -w^* - \dfrac{\eta\sigma_1(\eta^*-\xi^*)}{\mathbb{X}} \end{bmatrix} \begin{bmatrix} \mathbb{X}_0 & 0 \\ 0 & \mathbb{X} \end{bmatrix} + \\ &\quad + \begin{bmatrix} \mathbb{X}_0 & 0 \\ 0 & \mathbb{X} \end{bmatrix} \begin{bmatrix} A_0^* & -\mathbb{X}_0^{-1}B_0\sigma_1\eta^* \\ \dfrac{\xi\sigma_1 B_0^*}{\mathbb{X}} & -w - \dfrac{(\eta^*-\xi^*)\sigma_1\eta^*}{\mathbb{X}} \end{bmatrix} \\ &= \begin{bmatrix} A_0\mathbb{X}_0 + \mathbb{X}_0 A_0^* & -B_0\sigma_1(\eta^*-\xi^*) \\ -(\eta-\xi)\sigma_1 B_0^* & -\mathbb{X}(w+w^*) - \eta\sigma_1(\eta^*-\xi^*) - (\eta^*-\xi^*)\sigma_1\eta^* \end{bmatrix} \\ &= \begin{bmatrix} A_0\mathbb{X}_0 + \mathbb{X}_0 A_0^* & -B_0\sigma_1(\eta^*-\xi^*) \\ -(\eta-\xi)\sigma_1 B_0^* & \eta\sigma_1\eta^* - \xi\sigma_1\xi^* - \eta\sigma_1(\eta^*-\xi^*) - (\eta^*-\xi^*)\sigma_1\eta^* \end{bmatrix} \\ &= -B_S\sigma_1 B_S^*. \end{aligned}$$

The last equality follows easily from the Lyapunov equation for the given realization of S_0 and the formula (3.13). $\square$

4. The tangential Schur algorithm in the class $\boldsymbol{\mathcal{SI}}$

Our strategy to the tangential Schur algorithm in the class $\boldsymbol{\mathcal{SI}}$ relies on the following theorem. This theorem shows in particular that one cannot use the naive approach of applying the classical tangential Schur algorithm for each t_2 (that is, looking at t_2 as a mere parameter).

Theorem 4.1. *Let us fix the parameters σ_1, σ_2, and γ, and the interval I. Then for every $t_2^0 \in \mathrm{I}$ there is a one-to-one correspondence between pairs (γ_*, S) such that $S \in \boldsymbol{\mathcal{SI}}$ and γ_* continuous in a neighborhood of t_2^0, and functions $Y(\lambda) \in \boldsymbol{\mathcal{S}}(t_2^0)$.*

Proof: Let ϕ be the map which to a pair (γ_*, S) associates the function $S(\lambda, t_2^0) \in \boldsymbol{\mathcal{S}}(t_2^0)$. The converse map $\psi : Y(\lambda) \to Y(\lambda, t_2)$ was introduced in [M], [MV1, chapter 7] in a more general setting, and is defined as follows. Suppose that we have realized the transfer function $Y(\lambda)$ in the form

$$Y(\lambda) = I_p - B_0^* \mathbb{X}_0^{-1} (\lambda I - A_1)^{-1} B_0 \sigma_1(t_2^0).$$

Then we construct $B(t_2)$ from the differential equation with the spectral matrix parameter A_1:

$$\frac{d}{dt_2}[B(t_2)\sigma_1(t_2)] + A_1 B(t_2)\sigma_2(t_2) + B(t_2)\gamma(t_2) = 0, \quad B(t_2^0) = B_0. \tag{4.1}$$

In fact, the function $B(t_2)$ is given by the formula

$$B(t_2) = \oint (\lambda I - A_1)^{-1} B_0 \sigma_1^{-1} \Phi^{-1}(\lambda, t_2, t_2^0) d\lambda. \tag{4.2}$$

Next we construct $\mathbb{X}(t_2)$ on the maximal interval I, where it is invertible (2.14) via the formula

$$\frac{d}{dt_2}\mathbb{X}(t_2) = B(t_2)\sigma_2(t_2)B(t_2)^*, \quad \mathbb{X}(t_2^0) = \mathbb{X}_0. \tag{4.3}$$

Finally, we define

$$\begin{aligned}\gamma_*(t_2) \ &= \gamma(t_2) + \sigma_2(t_2)B(t_2)^*\mathbb{X}^{-1}(t_2)B(t_2)\sigma_1(t_2) - \\ &\quad - \sigma_1(t_2)B(t_2)^*\mathbb{X}^{-1}(t_2)B(t_2)\sigma_2(t_2).\end{aligned}$$

Then easy computations show that the function

$$S(\lambda, t_2) = I_p - B(t_2)^*\mathbb{X}^{-1}(t_2)(\lambda I - A_1)^{-1}B(t_2)\sigma_1(t_2) \tag{4.4}$$

is in the class $\boldsymbol{\mathcal{CI}}$ corresponding to the parameters $\sigma_1(t_2)$,$\sigma_2(t_2)$,$\gamma(t_2)$, and $\gamma_*(t_2)$ for $t_2 \in \mathrm{I}$. **????????????????? Moreover, the Lyapunov equation** (3.8) **holds for every** $t_2 \in \mathrm{I}$**, and thus the realization** (4.4) **is minimal for every** $t_2 \in \mathrm{I}$.

Note that the composition $\phi \circ \psi = id$, since starting from a function $Y \in \boldsymbol{\mathcal{S}}(t_2^0)$, constructing $Y(\lambda, t_2)$ and taking its value at t_2^0, we shall obtain again Y from the initial conditions of the differential equations (4.1) and (4.3) defining $B(t_2), \mathbb{X}(t_2)$.

In order to show that $\psi \circ \phi = id$, we start from a function $S(\lambda, t_2) \in \boldsymbol{\mathcal{SI}}$ and take its value $S(\lambda, t_2^0) \in \boldsymbol{\mathcal{S}}(t_2^0)$. Using the construction above, we shall obtain a function $Y(\lambda, t_2)$. Note that the two functions $S(\lambda, t_2)$ and $Y(\lambda, t_2)$ have the same value at t_2^0 and maps solutions of the same input LDE to (possibly different) output LDEs, i.e. :

$$\begin{aligned} S(\lambda, t_2) = \Phi_*(\lambda, t_2, t_2^0) S(\lambda, t_2^0) \Phi^{-1}(\lambda, t_2, t_2^0), \\ Y(\lambda, t_2) = \Phi'_*(\lambda, t_2, t_2^0) S(\lambda, t_2^0) \Phi^{-1}(\lambda, t_2, t_2^0). \end{aligned}$$

Then the function $S^{-1}(\lambda, t_2) Y(\lambda, t_2)$ is equal to I_p at infinity and is entire. By Liouville's theorem it is a constant function and is equal to I_p. Thus

$$\Phi_*(\lambda, t_2, t_2^0) = \Phi'_*(\lambda, t_2, t_2^0),$$

from where we obtain that

$$\Phi_*^{-1}(\lambda, t_2, t_2^0) \Phi'_*(\lambda, t_2, t_2^0) = I_p.$$

Differentiating both sides of this last equation we get to

$$\begin{aligned} 0 &= \frac{\partial}{\partial t_2} [\Phi_*^{-1}(\lambda, t_2, t_2^0) \Phi'_*(\lambda, t_2, t_2^0)] = \\ &= \Phi_*^{-1}(\lambda, t_2, t_2^0) \sigma_1^{-1}(t_2)(-\gamma(t_2) + \gamma'(t_2)) \Phi'_*(\lambda, t_2, t_2^0). \end{aligned}$$

Since the matrices $\Phi_*(\lambda, t_2, t_2^0), \Phi'_*(\lambda, t_2, t_2^0), \sigma_1(t_2)$ are invertible we obtain that $\gamma(t_2) = \gamma'(t_2)$. □

Remark: Last theorems claims that the correspondence is between "initial" values $S(\lambda, t_2^0)$ and pairs $(S(\lambda, t_2), \gamma_*(t_2))$. Notice that it is possible to obtain functions with the same $\gamma_*(t_2)$ with different initial conditions:

Proposition 4.2. *Suppose that there exists a function $Y \in \boldsymbol{\mathcal{S}}(t_2^0)$, which commutes with $\Phi(\lambda, t_2, t_2^0)$ and suppose that a function $S \in \boldsymbol{\mathcal{SI}}$ corresponds to certain vessel parameters $\sigma_1, \sigma_2, \gamma, \gamma_*$. Then the function SY belongs to the class $\boldsymbol{\mathcal{SI}}$ and corresponds to the same vessel parameters $\sigma_1, \sigma_2, \gamma, \gamma_*$.*

Proof: Using formula 2.21 we obtain that

$$S(\lambda, t_2) = \Phi_*(\lambda, t_2, t_2^0) S(\lambda, t_2^0) \Phi^{-1}(\lambda, t_2, t_2^0).$$

Consequently,

$$S(\lambda, t_2)Y(\lambda) = \Phi_*(\lambda, t_2, t_2^0)S(\lambda, t_2^0)Y(\lambda)\Phi^{-1}(\lambda, t_2, t_2^0)$$

intertwines solutions of the input (2.16) and the output (2.17) ODEs with the spectral parameter λ, and is identity at infinity, because S and Y and their product are such. Thus by the definition the function $SY \in \boldsymbol{\mathcal{SI}}$ and corresponds to the same spectral parameters as S. $\square$
For example (to be studied in subsection 5.5), taking Sturm-Liouville (SL) vessel parameters

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \sigma_2 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \gamma = \begin{bmatrix} 0 & 0 \\ 0 & i \end{bmatrix}$$

we shall obtain that

$$\Phi(\lambda, t_2, t_2^0) = V \begin{bmatrix} e^{-k(t_2-t_2^0)} & 0 \\ 0 & e^{k(t_2-t_2^0)} \end{bmatrix} V^{-1},$$

where

$$V = \begin{bmatrix} -\sqrt{\dfrac{i}{\lambda}} & \sqrt{\dfrac{i}{\lambda}} \\ 1 & 1 \end{bmatrix}, \quad k = \sqrt{i\lambda}.$$

Taking Y, which commutes with $\sigma_1^{-1}(\sigma_2\lambda + \gamma) = \begin{bmatrix} 0 & i \\ \lambda & 0 \end{bmatrix}$, i.e. of the form

$$Y(\lambda) = I_2 - \begin{bmatrix} a(\lambda) & \dfrac{ic(\lambda)}{\lambda} \\ c(\lambda) & a(\lambda) \end{bmatrix} \tag{4.5}$$

for functions $a(\lambda), c(\lambda)$, which are zero at infinity, we shall obtain that for any $S \in \boldsymbol{\mathcal{SI}}$, the function $SY \in \boldsymbol{\mathcal{SI}}$ and corresponds to the same vessel parameters. But in this case (which can be also generalized) it turns to be a necessary condition as the following theorem states:

Lemma 4.3. *Given SL vessel parameters $\sigma_1, \sigma_2, \gamma, \gamma_*(x)$, there exists a unique initial value $S(0, \lambda)$ up to a scalar t_2-independent symmetric and identity at infinity function.*

Proof: For this it is enough to prove that if a function $Y(\lambda, t_2) \in \boldsymbol{\mathcal{SI}}$ intertwines solutions of the input LDE (2.16) with itself, then $Y(\lambda, t_2) = Y(\lambda, t_2^0)$, i.e is a constant function, which moreover commutes with $\Phi_I(\lambda, t_2, t_2^0)$. Indeed, if we are given two functions $S_1(\lambda, t_2)$, $S_2(\lambda, t_2)$, then the function $S_1^{-1}(\lambda, t_2)S_2(\lambda, t_2)$ will intertwine solutions of the input LDE with itself and as a result is constant $Y(\lambda) \in \boldsymbol{\mathcal{S}}$, which is moreover commutes with $\Phi_I(\lambda, t_2)$. Following the paragraph preceding the theorem, $Y(\lambda) = I$ and this means that $S_1(\lambda, t_1^0) = S_2(\lambda, t_2^0)$.

Let us show first that if $S(\lambda, t_2)$ intertwines solutions of the input LDE (2.16) with itself and is identity at $\lambda = \infty$, then it is a constant matrix, which commutes with $\Phi(\lambda, t_2, t_2^0)$. Performing simple calculations, we can find that

$$\sigma_1^{-1}(\sigma_2\lambda+\gamma) = \begin{bmatrix} 0 & i \\ \lambda & 0 \end{bmatrix}, \quad \Phi(\lambda, t_2, t_2^0) = V \begin{bmatrix} e^{-k(t_2-t_2^0)} & 0 \\ 0 & e^{k(t_2-t_2^0)} \end{bmatrix} V^{-1},$$

where

$$V = \begin{bmatrix} -\sqrt{\dfrac{i}{\lambda}} & \sqrt{\dfrac{i}{\lambda}} \\ 1 & 1 \end{bmatrix}, \quad k = \sqrt{i\lambda}.$$

Consequently, for the expression $S(\lambda, t_2) = \Phi(\lambda, t_2, t_2^0)S_0(\lambda)\Phi^{-1}(\lambda, t_2, t_2^0)$ to be identity for $\lambda = \infty$, it is necessary to "cancel" the essential singularity arising from two entire functions $\Phi(\lambda, t_2, t_2^0)$ and $\Phi^{-1}(\lambda, t_2, t_2^0)$ (or more precisely, let them cancel each other).
Using the formula for $\Phi(\lambda, t_2, t_2^0)$

$$\Phi(\lambda, t_2, t_2^0)S_0(\lambda, t_2^0)\Phi^{-1}(\lambda, t_2, t_2^0) =$$
$$V \begin{bmatrix} e^{-k(t_2-t_2^0)} & 0 \\ 0 & e^{k(t_2-t_2^0)} \end{bmatrix} V^{-1}S(\lambda, t_2^0)V^{-1} \begin{bmatrix} e^{k(t_2-t_2^0)} & 0 \\ 0 & e^{-k(t_2-t_2^0)} \end{bmatrix} V^{-1} = I$$

and considering coefficients of the exponents, we may conclude that in order to cancel the exponents it is necessary to demand

$$V^{-1}S(\lambda, t_2^0)V = \begin{bmatrix} b(\lambda) & 0 \\ 0 & d(\lambda) \end{bmatrix}.$$

for some analytic in $\sqrt{\lambda}$ at infinity functions $b(\lambda), d(\lambda)$ which are 1 there. From here it follows that

$$S(\lambda, t_2^0) = \begin{bmatrix} -\sqrt{\dfrac{i}{\lambda}}(b(\lambda) + d(\lambda)) & \dfrac{i}{\lambda}(d(\lambda) - b(\lambda)) \\ d(\lambda) - b(\lambda) & -\sqrt{\dfrac{i}{\lambda}}(b(\lambda) + d(\lambda)) \end{bmatrix}.$$

denoting here $1 - a(\lambda) = -\sqrt{\dfrac{i}{\lambda}}(b(\lambda) + d(\lambda))$ and $-c(\lambda) = d(\lambda) - b(\lambda)$, we shall obtain that $S_0(\lambda)$ is of the form (4.5), i.e. commutes with the fundamental matrix $\Phi(\lambda, t_2, t_2^0)$.
Finally, let us consider a function of this form and will require the symmetry condition for it:

$$S^*(\lambda, t_2^0)\sigma_1 S(\lambda, t_2^0) = \sigma_1.$$

Plugging here the expression (4.5) for $S(\lambda, t_2^0)$

$$S(\lambda, t_2^0) = I_2 - \begin{bmatrix} a(\lambda) & \frac{ic(\lambda)}{\lambda} \\ c(\lambda) & a(\lambda) \end{bmatrix}$$

we shall obtain that on the one imaginary axis, where $\lambda = -\bar{\lambda}$, the following system of equations must hold

$$\begin{cases} (1 - a^*(-\bar{\lambda}))c(\lambda) + c^*(-\bar{\lambda})(1 - a(\lambda)) = 0, \\ (1 - a^*(-\bar{\lambda}))(1 - a(\lambda)) + \frac{1}{\lambda} c^*(-\bar{\lambda})c(\lambda) = 1, \\ (1 - a^*(-\bar{\lambda}))(1 - a(\lambda)) + \frac{1}{\bar{\lambda}} c^*(-\bar{\lambda})c(\lambda) = 1, \\ \frac{ic^*(-\bar{\lambda})(1 - a(\lambda))}{\lambda} + \frac{ic(\lambda)(1 - a^*(-\bar{\lambda}))}{\lambda} = 0. \end{cases}$$

Subtracting the second and the third equation we obtain that (remember that $\lambda = -\bar{\lambda}$)

$$c^*(-\bar{\lambda})c(\lambda)(\frac{1}{\lambda} - \frac{1}{\bar{\lambda}}) = c^*(-\bar{\lambda})c(\lambda)\frac{2}{\lambda} = 0.$$

from where it follows that on the imaginary axis at the points of analyticity of $c(\lambda)$ it holds that $c(\lambda) = 0$. Since it is analytic there, it will be globally zero. Notice that the last system of equations for the trivial $c(\lambda)$ gives

$$(1 - a^*(-\bar{\lambda}))(1 - a(\lambda)) = 1,$$

Which means that $1 - a(\lambda)$ is of norm one on the imaginary axis. □

We now focus on the rational case. We first note that starting from different realizations at t_2^0 we shall obtain the same $\gamma_*(t_2)$. More precisely we have the following theorem:

Theorem 4.4. *Suppose that there are two minimal realizations of the function $S(\lambda, t_2^0) \in \boldsymbol{\mathcal{RS}}$:*

$$S(\lambda, t_2^0) = I_p - B_\ell^* \mathbb{X}_\ell^{-1}(\lambda I - A_1)^{-1} B_\ell \sigma_1(t_2^0), \quad \ell = 1, 2,$$

with associated similarity matrix V. Then the functions

$$S_\ell(\lambda, t_2) = I_p - B_\ell(t_2)^* \mathbb{X}_\ell^{-1}(t_2)(\lambda I - A_\ell)^{-1} B_\ell(t_2)\sigma_1(t_2), \quad \ell = 1, 2,$$

obtained from these realizations via the construction in Theorem 4.1 are the same, and this holds if and only if:

$$B_2(t_2) = V B_1(t_2), \tag{4.6}$$

where t_2 varies in a neighborhood of t_2^0. Moreover, in this case the following formula holds

$$\mathbb{X}_2(t_2) = V \mathbb{X}_1(t_2) V^*.$$

Proof: Equality of the two realizations means that there exists an invertible matrix V such that

$$A_2 = VA_1V^{-1},\ B_2\sigma_1(t_2^0) = VB_1\sigma_1(t_2^0),\ B_1^*\mathbb{X}_1^{-1}V = B_2^*\mathbb{X}_2^{-1},$$

from which follows (see [AG, Lemma 2.1 p. 184] for instance) that

$$\mathbb{X}_2 = V\mathbb{X}_1V^*.$$

By the construction described in Theorem 4.1, the function $B_1(t_2)$ will satisfy (4.1)

$$\frac{d}{dt_2}[B_1(t_2)\sigma_1(t_2)] + A_1B_1(t_2)\sigma_2(t_2) + B_1(t_2)\gamma(t_2) = 0, \quad B_1(t_2^0) = B_1,$$

and the function $B_2(t_2)$ will satisfy the same equation with A_2 instead of A_1:

$$\frac{d}{dt_2}[B_2(t_2)\sigma_1(t_2)] + A_2B_2(t_2)\sigma_2(t_2) + B_2(t_2)\gamma(t_2) = 0, \quad B_2(t_2^0) = B_2.$$

Using the equalities $A_2 = VA_1V^{-1}$, $B_2 = VB_1$ we obtain that the function $V^{-1}B_2(t_2)$ satisfies

$$\begin{array}{l} \frac{d}{dt_2}[V^{-1}B_2(t_2)\sigma_1(t_2)] + A_1V^{-1}B_2(t_2)\sigma_2(t_2) + V^{-1}B_2(t_2)\gamma(t_2) = 0, \\ V^{-1}B_2(t_2^0) = B_1, \end{array}$$

which is the same differential equation as for $B_1(t_2)$. Thus $B_2(t_2) = VB_1(t_2)$. Similarly, considering the differential equations

$$\frac{d}{dt_2}\mathbb{X}_i(t_2) = B_i(t_2)\sigma_1B_i(t_2)^*, \quad \mathbb{X}_i(t_2^0) = \mathbb{X}_i, \quad i = 1, 2,$$

we obtain that

$$\mathbb{X}_2(t_2) = V\mathbb{X}_1(t_2)V^*.$$

Consequently,

$$\begin{aligned} B_2^*(t_2)\mathbb{X}_2^{-1}(t_2)B(t_2) &= B_1^*(t_2)V^*V^{-1*}\mathbb{X}_1^{-1}(t_2)V^{-1}VB_1(t_2) \\ &= B_1^*(t_2)\mathbb{X}_1^{-1}(t_2)B_1(t_2), \end{aligned}$$

from where we conclude that the same function γ_* is associated to $S_1(\lambda, t_2)$ and $S_2(\lambda, t_2)$. Since these functions coincide for $t_2 = t_2^0$ and map the same input ODE, we obtain that they are equal in a neighborhood of t_2^0. $\square$

The following notion has been introduced and studied in [M2].

Definition 4.5. *Let $S \in \mathcal{RSI}$ with a realization* (4.4)*. The function*

$$\tau(t_2) = \det \mathbb{X}(t_2).$$

is called the τ-function associated to S.

It follows from Theorem 4.4 that the τ function is well defined up to a multiplicative strictly positive constant. Indeed using the notation of the theorem,

$$\det \mathbb{X}_2(t_2) = \det[V\mathbb{X}_1(t_2)V^*] = \det \mathbb{X}_1(t_2)\det(VV^*).$$

We now introduce the counterpart of the tangential Schur algorithm in the class $\boldsymbol{\mathcal{RSI}}$. Let $S \in \boldsymbol{\mathcal{RSI}}$ and fix $t_2^0 \in \mathrm{I}$. The function $S(\lambda, t_2^0) \in \boldsymbol{\mathcal{RS}}$. Consider now a space $\mathcal{M}$ with $\mathbb{X} > 0$ and corresponding function Θ as in Corollary 3.3. This is always possible in view of Lemma 3.4. It follows from Theorem 3.2 that there exists $S_0 \in \boldsymbol{\mathcal{RS}}$ such that

$$S(\lambda, t_2^0) = T_{\Theta(\lambda)}(S_0(\lambda)). \tag{4.7}$$

Applying Theorem 4.1 to S_0, we obtain a uniquely defined function $S_0(\lambda, t_2) \in \boldsymbol{\mathcal{RSI}}$, such that at t_2^0 the relation (4.7) holds.

Definition 4.6. *The map* T_{Θ,t_2^0}

$$S(\lambda, t_2) \mapsto S_0(\lambda, t_2)$$

is the time-varying counterpart of the linear fractional transformation (4.7)*. We will call it a generalized linear fractional transformation.*

If $S(\lambda, t_2) \in \boldsymbol{\mathcal{SI}}$ corresponds to the vessel parameters σ_1, σ_2, γ, γ_*, then $S_0(\lambda, t_2^0) \in \boldsymbol{\mathcal{SI}}$ corresponds to the vessel parameters σ_1, σ_2, γ, $\gamma_{0,*}$ for a uniquely defined function $\gamma_{0,*}(t_2)$. Moreover, for $t_2 = t_2^0$ we have the usual linear fractional transformation (4.7).

As a consequence of Theorem 4.1 the following lemma holds.

Lemma 4.7. *For a given J-inner function Θ and a given point t_2^0, the map T_{Θ,t_2^0} is one-to-one from $\boldsymbol{\mathcal{RS}}$ into $\boldsymbol{\mathcal{RSI}}$.*

Proof: Notice that for a given t_2^0 the map T_Θ is injective. Furthermore, using Theorem 4.1, every $S \in \boldsymbol{\mathcal{RSI}}$ is uniquely defined by the function $S(\lambda, t_2^0) \in \boldsymbol{\mathcal{RS}}$. The result follows. $\square$

Suppose now that we start from $S_0(\lambda) \equiv I_p$ and apply n linear fractional transformations for a fixed t_2^0, using the data $< w_i, \xi_i, \eta_i >$ $(i = 1, \ldots n)$ to construct the corresponding J-inner functions. We obtain a function

$$S_n(\lambda) = T_\Theta(I) = I - B_n^* X_n^{-1}(\lambda I - A_n)^{-1} B_n \sigma_1 \in \boldsymbol{\mathcal{RS}}.$$

Using iteratively formulas (3.15), (3.16), (3.17) we obtain that

$$B_n = \begin{bmatrix} \eta_1 - \xi_1 \\ \vdots \\ \eta_n - \xi_n \end{bmatrix},$$

$$\mathbb{X}_n = \mathrm{diag}[\widetilde{\mathbb{X}}_1, \ldots, \widetilde{\mathbb{X}}_n],$$

and

(4.8)

$$A_n = \begin{bmatrix} -w_1^* - \dfrac{\eta_1\sigma_1(\eta_1^* - \xi_1^*)}{\widetilde{\mathbb{X}}_1} & \dfrac{(\eta_1 - \xi_1)\sigma_1\xi_2^*}{\widetilde{\mathbb{X}}_2} & \cdots & \dfrac{(\eta_1 - \xi_1)\sigma_1\xi_n^*}{\widetilde{\mathbb{X}}_n} \\ -\dfrac{\eta_2\sigma_1(\eta_1^* - \xi_1^*)}{\widetilde{\mathbb{X}}_1} & -w_2^* - \dfrac{\eta_2\sigma_1(\eta_2^* - \xi_2^*)}{\widetilde{\mathbb{X}}_2} & \cdots & \dfrac{(\eta_2 - \xi_2)\sigma_1\xi_n^*}{\widetilde{\mathbb{X}}_n} \\ \vdots & \vdots & \ddots & \vdots \\ -\dfrac{\eta_n\sigma_1(\eta_1^* - \xi_1^*)}{\widetilde{\mathbb{X}}_1} & -\dfrac{\eta_n\sigma_1(\eta_2^* - \xi_2^*)}{\widetilde{\mathbb{X}}_2} & \cdots & -w_n^* - \dfrac{\eta_n\sigma_1(\eta_n^* - \xi_n^*)}{\widetilde{\mathbb{X}}_n} \end{bmatrix},$$

where $\widetilde{\mathbb{X}}_i$ is defined by (3.13):

$$\widetilde{\mathbb{X}}_i = \frac{\xi_i\sigma_1\xi_i^* - \eta_i\sigma_1\eta_i^*}{w_i + w_i^*}, \quad i = 1, \ldots n.$$

Assume now that, starting from the identity matrix we apply this procedure for two different sets of data (with the same n) at two different points t_2^1 and t_2^2. The following theorem answers the question as when we obtain the same function, that is, when do we have:

$$T_{\Theta_1, t_2^1}(I_p) = T_{\Theta_2, t_2^2}(I_p).$$

Theorem 4.8. *Suppose that there are given two sets of n triples $< w_i^1, \xi_i^1, \eta_i^1 >$ and $< w_i^2, \xi_i^2, \eta_i^2 >$, with corresponding $\Theta_\ell, \ell = 1, 2$. Then necessary and sufficient conditions for equality of the two functions*

$$S_\ell(\lambda, t_2) = T_{\Theta_\ell, t_2^\ell}(I) = I_p - B_n^\ell(\mathbb{X}_n^\ell)^{-1}(\lambda I - A_n^\ell)^{-1}B_n^\ell\sigma_1, \quad \ell = 1, 2$$

are:

(1) *The corresponding matrices A_n^1 and A_n^2 defined by (4.8) are similar, i.e. there exists an invertible matrix V such that $A_n^1 = VA_n^2V^{-1}$,*
(2) $\oint(\lambda I - A_n^1)^{-1}B_n^1\sigma_1^{-1}\Phi^{-1}(\lambda, t_2^2, t_2^1)d\lambda = VB_n^2$.

Proof: From Theorem 4.1, a necessary and sufficient condition for the functions to be equal is that

$$S_1(\lambda, t_2^2) = S_2(\lambda, t_2^2).$$

From 4.4 this holds if and only if

$$A_n^2 = VA_n^1V^{-1},\ B_n^2(t_2^2) = VB_n^1,\ \mathbb{X}_n^2(t_2^2) = V\mathbb{X}_n^1V^*$$

for a uniquely defined invertible matrix V. The result follows using formula (4.2). □

A more general construction in this setting is obtained if one supposes that at each step different values of t_2 are chosen. In this case the construction of the function $S_n(\lambda, t_2)$ is more complicated, and can be computed recursively. The formulas are very involved in this case and we can see no real advantage to develop them at this point.

5. Markov moments and partial realization problem in the class $\boldsymbol{\mathcal{SI}}$

Let $S \in \boldsymbol{\mathcal{S}}$, and consider the Laurent expansion at infinity

$$S(\lambda) = I_p - B^*\mathbb{X}^{-1}(\lambda I - A)^{-1}B\sigma_1 = I_p - \sum_{i=0}^{\infty} \frac{1}{\lambda^{i+1}} B^*\mathbb{X}^{-1}A^iB\sigma_1$$

in terms of a given minimal realization. The matrices $H_i = B^*\mathbb{X}^{-1}A^iB\sigma_1$ are called the *Markov moments* of S. The Partial Realization Problem (or moment problem at infinity) in $\boldsymbol{\mathcal{S}}$ is defined as follows: Given the first $n+1$ Markov moments $H_0, \dots, H_n$, find all functions (if any) $S \in \boldsymbol{\mathcal{S}}$ with these first $n+1$ moments. See [GKL] for a general study of the partial realization problem. Similarly, one can define the Markov moments for an element $S \in \boldsymbol{\mathcal{SI}}$. We now give necessary conditions which the moments of a function $S \in \boldsymbol{\mathcal{SI}}$ have to satisfy.

It is important to notice the following: fixing $t_2 = t_2^0$ and solving the corresponding classical moment problem will not lead to a solution of the problem in the class $\boldsymbol{\mathcal{SI}}$ because we cannot obtain the function γ (and hence γ_*) from this solution.

5.1. Fundamental properties of $S(\lambda, t_2)$.

Before we consider moments of the function $S(\lambda, t_2)$ we present some of its fundamental properties which shed more light on the deriving of the moment equations.

Theorem 5.1. *For fixed $\sigma_1, \sigma_2, \gamma$, a necessary condition on γ_* so that there exists a vessel with the vessel parameters $\sigma_1, \sigma_2, \gamma, \gamma_*$ is*

$$\det \Phi_*(\lambda, t_2, t_2^0) = \det \Phi(\lambda, t_2, t_2^0)$$

Proof: Let $S \in \boldsymbol{\mathcal{SI}}$ be a function corresponding to the parameters $\sigma_1, \sigma_2, \gamma, \gamma_*$. Then using a realization (4.4)

$$S(\lambda, t_2) = I_p - B(t_2)^*\mathbb{X}^{-1}(t_2)(\lambda I - A_1)^{-1}B(t_2)\sigma_1(t_2)$$

and Lyapunov equation (3.8) we shall obtain that

$$\begin{aligned}
\det S(\lambda,t_2) &= \det\big(I_p - B(t_2)^*\mathbb{X}^{-1}(t_2)(\lambda I - A_1)^{-1}B(t_2)\sigma_1(t_2)\big)\\
&= \det\big(I - B(t_2)\sigma_1(t_2)B(t_2)^*\mathbb{X}^{-1}(t_2)(\lambda I - A_1)^{-1}\big)\\
&= \det\big(I + (A_1\mathbb{X}(t_2)+\mathbb{X}(t_2)A_1^*)\mathbb{X}^{-1}(t_2)(\lambda I - A_1)^{-1}\big)\\
&= \det\big(I + A_1(\lambda I - A_1)^{-1} + \mathbb{X}(t_2)A_1^*\mathbb{X}^{-1}(t_2)(\lambda I - A_1)^{-1}\big)\\
&= \det\left((\lambda I + \mathbb{X}(t_2)A_1^*\mathbb{X}^{-1}(t_2))(\lambda I - A_1)^{-1}\right)\\
&= \det(\lambda I + A_1^*)\det(\lambda I - A_1)^{-1}\\
&= \det S(\lambda, t_2^0).
\end{aligned}$$

Consequently, taking determinant of the formula (2.21)

$$S(\lambda,t_2)\Phi(\lambda,t_2,t_2^0) = \Phi_*(\lambda,t_2,t_2^0)S(\lambda,t_2^0)$$

we shall obtain that $\det\Phi_*(\lambda,t_2,t_2^0) = \det\Phi(\lambda,t_2,t_2^0)$ for all points λ, where $\det S(\lambda,t_2^0)$ exists and is different from zero. Since it happens for all points outside the spectrum of A_1 and the functions $\det\Phi_*(\lambda,t_2,t_2^0)$, $\det\Phi(\lambda,t_2,t_2^0)$ are entire they are equal for all λ. $\square$

Next theorem put some light on the connection between equations, which determine the transfer function $S(\lambda,t_2)$ and its moments. This theorem is similar to the property of a solution of a Riccati equation [Ze, theorem 2.1]

Theorem 5.2. *Suppose that $S(\lambda,t_2)$ is a continuous function of t_2 for each λ, meromorphic in λ for each t_2 and satisfies $S(\infty,t_2)=I$. Suppose also that $S(\lambda,t_2)$ is an intertwining function of LDEs* (2.16) *and* (2.17). *Then if the symmetry condition* (2.23)

$$S(\lambda,t_2) = \sigma_1^{-1}(t_2)S^{-1*}(-\bar\lambda,t_2)\sigma_1(t_2)$$

holds for t_2^0, then it holds for all t_2.

Proof: Since $S(\lambda,t_2)$ intertwines solutions of (2.16) and (2.17), then it satisfies the differential equation (2.22)

$$\begin{aligned}
\frac{\partial}{\partial t_2}S(\lambda,t_2) = \sigma_1^{-1}(t_2)(\sigma_2(t_2)\lambda + \gamma_*(t_2))S(\lambda,t_2) - \\
-S(\lambda,t_2)\sigma_1^{-1}(t_2)(\sigma_2(t_2)\lambda+\gamma(t_2)).
\end{aligned}$$

Consequently, using properties of γ_*,γ appearing in definition 1.1 we obtain that the function $\sigma_1^{-1}(t_2)S^{-1*}(-\bar\lambda,t_2)\sigma_1(t_2)$ satisfies the same differential equation. If these two functions are equal at t_2^0, from the uniqueness of solution for a differential equation with continuous coefficients, they are also equal for all t_2. $\square$

5.2. **Restrictions on Markov moments for functions in $\boldsymbol{\mathcal{SI}}$.** We study the Markov moments of a function $S \in \boldsymbol{\mathcal{SI}}$, which maps solutions of the input ODE (2.16) to the output ODE (2.17) using a minimal realization (4.4) of S.
If A_1 is a constant matrix, at some stage the elements $I, A_1, \ldots, A_1^n$ will be linearly dependent and we obtain:

Lemma 5.3. *Given $S \in \boldsymbol{\mathcal{RSI}}$, with Markov moments $H_i(t_2)$, $i = 0, 1, \ldots$. Then exists N and constants μ_j, $j = 1, \ldots, N$ such that*

$$\sum_{j=0}^{N+1} \mu_j H_{n-j}(t_2) = 0, \quad n \geq N+1. \tag{5.1}$$

Next we present formulas, which hold for the moments of a function in $\boldsymbol{\mathcal{SI}}$. They can be easily derived from theorem 5.2. We notice that the first moment $H_0(t_2)$ satisfies the linkage condition (1.3):

$$\gamma_*(t_2) - \gamma(t_2) = \sigma_2(t_2) H_0(t_2) - \sigma_1(t_2) H_0(t_2) \sigma_1^{-1}(t_2) \sigma_2(t_2).$$

Let us denote

$$H_0(t_2) = C(t_2) B(t_2) \sigma_1(t_2) = (B(t_2))^* X^{-1}(t_2) B(t_2) \sigma_1(t_2).$$

The functions $B(t_2), C(t_2)$ satisfy the following differential equations

$$\frac{d}{dt_2}[B(t_2)\sigma_1(t_2)] + A_1 B(t_2) \sigma_2(t_2) + B(t_2) \gamma(t_2) = 0 \tag{5.2}$$

$$\sigma_1(t_2) \frac{d}{dt_2} C(t_2) - \sigma_2(t_2) C(t_2) A_1 - \gamma_*(t_2) C(t_2) = 0, \tag{5.3}$$

see [M, MV1]. Thus, differentiating $H_0(t_2)$, we obtain

$$\begin{aligned} \tfrac{d}{dt_2} H_0(t_2) &= \tfrac{d}{dt_2}[C(t_2) B(t_2) \sigma_1(t_2)] \\ &= \sigma_1^{-1}(t_2) \sigma_2(t_2) C(t_2) A_1 B(t_2) \sigma_1(t_2) - C(t_2) A_1 B(t_2) \sigma_2(t_2) + \\ &\quad + \sigma_1^{-1}(t_2) \gamma_*(t_2) H_0(t_2) - H_0(t_2) \sigma_1^{-1}(t_2) \gamma(t_2). \end{aligned}$$

In other words the second moment $H_1(t_2) = C(t_2) A_1 B(t_2) \sigma_1(t_2)$ satisfies the following differential equation

$$\begin{aligned} &\sigma_1^{-1}(t_2) \sigma_2(t_2) H_1(t_2) - H_1(t_2) \sigma_1^{-1}(t_2) \sigma_2(t_2) = \\ &= \tfrac{d}{dt_2} H_0(t_2) - \sigma_1^{-1} \gamma_* H_0(t_2) + H_0(t_2) \sigma_1^{-1}(t_2) \gamma(t_2). \end{aligned}$$

In the same manner the moments $H_i(t_2)$ and $H_{i+1}(t_2)$ are connected by the following differential equation

$$\begin{aligned} &\sigma_1^{-1}(t_2) \sigma_2(t_2) H_{i+1} - H_{i+1}(t_2) \sigma_1^{-1}(t_2) \sigma_2(t_2) = \\ &= \tfrac{d}{dt_2} H_i(t_2) - \sigma_1^{-1}(t_2) \gamma_*(t_2) H_i(t_2) + H_i(t_2) \sigma_1^{-1}(t_2) \gamma(t_2). \end{aligned} \tag{5.4}$$

Notice also that

$$\begin{aligned}
H_i\sigma_1^{-1}H_0^*\sigma_1 &= B^*X^{-1}A_1^iB\sigma_1B^*X^{-1}B\sigma_1 = \\
&= B^*X^{-1}A_1^i[-A_1X - XA_1^*]X^{-1}B\sigma_1 = \\
&= -H_{i+1} - B^*X^{-1}A_1^iXA_1^*X^{-1}B\sigma_1 = \\
&= -H_{i+1} - B^*X^{-1}A_1^{i-1}[-B\sigma_1B^* - XA_1^*]A_1^*X^{-1}B\sigma_1 = \\
&= -H_{i+1} + B^*X^{-1}A_1^{i-1}B\sigma_1B^*A_1^*X^{-1}B\sigma_1 + \\
&+ B^*X^{-1}A_1^{i-1}XA_1^*A_1^*X^{-1}B\sigma_1 = \\
&= -H_{i+1} + H_{i-1}\sigma_1^{-1}H_1^*\sigma_1 + B^*X^{-1}A_1^{i-1}XA_1^{2*}X^{-1}B\sigma_1 = \\
&= \cdots = \\
&= -H_{i+1} + H_{i-1}\sigma_1^{-1}H_1^*\sigma_1 - H_{i-2}\sigma_1^{-1}H_2\sigma_1 + \\
&+ \cdots + (-1)^i\sigma_1^{-1}H_{i+1}^*\sigma_1
\end{aligned}$$

Consequently, the following formula holds:

$$H_{i+1}\sigma_1^{-1} + (-1)^i\sigma_1^{-1}H_{i+1}^* = \sum_{j=0}^{i}(-1)^{j+1}H_{i-j}\sigma_1^{-1}H_j^*. \tag{5.5}$$

Remark: One can obtain the same equations (5.4) and (5.5) by taking the expansion of $S(\lambda, t_2)$ into Taylor series around $\lambda = \infty$ and equating the coefficients of each $\frac{1}{\lambda^i}$. More precisely, the algebraic condition (5.4) is a consequence of the symmetry condition (2.23) and the condition (5.5) is a result of the differential equation (2.22).

Finally, we show how the third condition in Proposition 2.5 is reflected in the moments $H_i(t_2)$. This condition means that the function $S(\lambda, t_2)$ is $\sigma_1(t_2)$ contractive, and, for example, the first moment $H_0(t_2)$ satisfies $H_0(t_2)\sigma_1^{-1}(t_2) > 0$. Using the minimality property, we have

$$\operatorname{span} A_1^n B\mathcal{E} = \mathcal{H}.$$

Thus, in order to have $\mathbb{X} > \epsilon I, \epsilon > 0$ (so that $\mathbb{X}^{-1}$ exists) it is enough to demand

$$\langle \mathbb{X}^{-1}\sum_{k=0}^{n} A^kBe_k, \sum_{k=0}^{n} A^kBe_k\rangle > \epsilon, \text{ for any choice of } e_k \in \mathcal{E}.$$

Thus we obtain that the following Pick matrix arises

$$\mathbb{P} = \begin{bmatrix} B^* \\ B^*A_1^* \\ \vdots \\ B^*(A_1^*)^n \end{bmatrix} \mathbb{X}^{-1} \begin{bmatrix} B & A_1B & \ldots & A_1^nB \end{bmatrix} > \epsilon I$$

which is equal to

$$\mathbb{P}_n = [p_{ij}]_{i,j}^n = [B^*(A_1^*)^i\mathbb{X}^{-1}A_1^jB] > \epsilon I, \quad n = 0, 1, 2, \ldots$$

Notice that the elements of the matrix are indexed from zero to n for convenience. Using Lyapunov equation (3.8) iteratively, we shall obtain that

$$\begin{aligned}
B^*(A_1^*)^m\mathbb{X}^{-1}A_1^nB &= B^*(A_1^*)^{m-1}(-\mathbb{X}^{-1}B\sigma_1B^*\mathbb{X}^{-1} - \mathbb{X}^{-1}A_1)A_1^nB \\
&= -\sigma_1^{-1}H_{m-1}^*\sigma_1H_n\sigma_1^{-1} - B^*(A_1^*)^{m-1}\mathbb{X}^{-1}A_1^{n+1}B \\
&= \cdots = \\
&= \sum_{k=0}^{m}(-1)^{k+1}\sigma_1^{-1}H_{m-1-k}^*\sigma_1H_{n+k}\sigma_1^{-1}, \quad m>0, \\
B^*\mathbb{X}^{-1}A_1^nB &= H_n.
\end{aligned}$$

So the Pick matrix is of the following form
(5.6)

$$\mathbb{P} = [p_{ij}]_{i,j=0}^n, \begin{cases} p_{ij} &= \sum_{k=0}^{i}(-1)^{k+1}\sigma_1^{-1}H_{i-1-k}^*\sigma_1H_{j+k}\sigma_1^{-1}, \quad i>1 \\ p_{0,j} &= H_j. \end{cases}$$

Next theorem shows that moments of a transfer function in $\boldsymbol{\mathcal{S}}$ satisfy recursive equations, involving algebraic and differential equations: The positivity of the matrix $\mathbb{P}$, in that case does not necessarily hold.

Lemma 5.4. *Denote the real and the imaginary parts of* $H_{2n+1}\sigma_1^{-1} = R_{2n+1} + iM_{2n+1}$. *Then if the moments* $H_0, \ldots, H_{2n}$ *satisfy the differential equations* (5.4) *and* (5.5), *then the real part* R_{2n+1} *satisfies the imaginary part of differential equation* (5.5):

$$\begin{aligned}
&2[\sigma_1^{-1}\sigma_2R_{2n+1} - R_{2n+1}\sigma_2\sigma_1^{-1}] = \\
&= \frac{d}{dt_2}[H_{2n}]\sigma_1^{-1} - \sigma_1^{-1}\gamma_*H_{2n}\sigma_1^{-1} + H_{2n}\sigma_1^{-1}\gamma\sigma_1^{-1} - \\
&-\sigma_1^{-1}\frac{d}{dt_2}[H_{2n}^*] + \sigma_1^{-1}H_{2n}^*\gamma_*^*\sigma_1^{-1} - \sigma_1^{-1}\gamma^*\sigma_1^{-1}H_{2n}^*.
\end{aligned} \tag{5.7}$$

Proof: Multiplying equation (5.4) by σ_1^{-1} on the right and using the notation $H_{2n+1}\sigma_1^{-1} = R_{2n+1} + iM_{2n+1}$ we shall obtain that

$$\begin{aligned}
&\sigma_1^{-1}\sigma_2(R_{2n+1} + iM_{2n+1}) - (R_{2n+1} + iM_{2n+1})\sigma_2\sigma_1^{-1} = \\
&\quad = \sigma_1^{-1}\sigma_2R_{2n+1} - R_{2n+1}\sigma_2\sigma_1^{-1} + i(\sigma_2\sigma_1^{-1}M_{2n+1} - M_{2n+1}\sigma_2\sigma_1^{-1}) = \\
&\quad\quad = \frac{d}{dt_2}[H_{2n}]\sigma_1^{-1} - \sigma_1^{-1}\gamma_*H_{2n}\sigma_1^{-1} + H_{2n}\sigma_1^{-1}\gamma\sigma_1^{-1}
\end{aligned}$$

Since $\sigma_1^{-1}\sigma_2 R_{2n+1} - R_{2n+1}\sigma_2\sigma_1^{-1}$ and $\sigma_2\sigma_1^{-1}M_{2n+1} - M_{2n+1}\sigma_2\sigma_1^{-1}$ are skew adjoint, last equation is equivalent to equality of real and imaginary parts of both sides:
(5.8)

$$2i[\sigma_1^{-1}\sigma_2 M_{2n+1} - M_{2n+1}\sigma_2\sigma_1^{-1}] = \frac{d}{dt_2}[H_{2n}]\sigma_1^{-1} + \sigma_1^{-1}\frac{d}{dt_2}[H_{2n}^*] +$$
$$+H_{2n}\sigma_1^{-1}\gamma\sigma_1^{-1} - \sigma_1^{-1}\gamma_* H_{2n}\sigma_1^{-1} - \sigma_1^{-1}H_{2n}^*\gamma_*^*\sigma_1^{-1} + \sigma_1^{-1}\gamma^*\sigma_1^{-1}H_{2n}^*,$$

and (5.7)

$$2[\sigma_1^{-1}\sigma_2 R_{2n+1} - R_{2n+1}\sigma_2\sigma_1^{-1}] = \frac{d}{dt_2}[H_{2n}]\sigma_1^{-1} - \sigma_1^{-1}\gamma_* H_{2n}\sigma_1^{-1} +$$
$$+H_{2n}\sigma_1^{-1}\gamma\sigma_1^{-1} - \sigma_1^{-1}\frac{d}{dt_2}[H_{2n}^*] + \sigma_1^{-1}H_{2n}^*\gamma_*^*\sigma_1^{-1} - \sigma_1^{-1}\gamma^*\sigma_1^{-1}H_{2n}^*.$$

Let us show that R_{2n+1}, defined in equation (5.5) also satisfies (5.7). Indeed,

$$2[\sigma_1^{-1}\sigma_2 R_{2n+1} - R_{2n+1}\sigma_2\sigma_1^{-1}] =$$
$$= \sigma_1^{-1}\sigma_2[\textstyle\sum_{j=0}^{2n}(-1)^{j+1}H_{2n-j}\sigma_1^{-1}H_j^*] - [\textstyle\sum_{j=0}^{2n}(-1)^{j+1}H_{2n-j}\sigma_1^{-1}H_j^*]\sigma_2\sigma_1^{-1} =$$
$$= \frac{d}{dt_2}[H_{2n}]\sigma_1^{-1} - \sigma_1^{-1}\gamma_* H_{2n}\sigma_1^{-1} + H_{2n}\sigma_1^{-1}\gamma\sigma_1^{-1} -$$
$$-\sigma_1^{-1}\frac{d}{dt_2}[H_{2n}^*] + \sigma_1^{-1}H_{2n}^*\gamma_*^*\sigma_1^{-1} - \sigma_1^{-1}\gamma^*\sigma_1^{-1}H_{2n}^*.$$

Performing rearrangement and using (5.5) for $i = 2n-1$, we shall obtain

$$\sigma_1^{-1}\sigma_2[\textstyle\sum_{j=0}^{2n}(-1)^{j+1}H_{2n-j}\sigma_1^{-1}H_j^*] - [\textstyle\sum_{j=0}^{2n}(-1)^{j+1}H_{2n-j}\sigma_1^{-1}H_j^*]\sigma_2\sigma_1^{-1} =$$
$$= \frac{d}{dt_2}[H_{2n}\sigma_1^{-1} - \sigma_1^{-1}H_{2n}^*] - \sigma_1^{-1}\gamma_* H_{2n}\sigma_1^{-1} + H_{2n}\sigma_1^{-1}\gamma\sigma_1^{-1} - H_{2n}\frac{d}{dt_2}\sigma_1^{-1} +$$
$$+\sigma_1^{-1}H_{2n}^*\gamma_*^*\sigma_1^{-1} - \sigma_1^{-1}\gamma^*\sigma_1^{-1}H_{2n}^* + \frac{d}{dt_2}[\sigma_1^{-1}]H_{2n}^* =$$
$$= \frac{d}{dt_2}[\textstyle\sum_{j=0}^{2n-1}(-1)^{j+1}H_{2n-1-j}\sigma_1^{-1}H_j^*] - \sigma_1^{-1}\gamma_* H_{2n}\sigma_1^{-1} + H_{2n}\sigma_1^{-1}\gamma\sigma_1^{-1} -$$
$$-H_{2n}\frac{d}{dt_2}\sigma_1^{-1} + \sigma_1^{-1}H_{2n}^*\gamma_*^*\sigma_1^{-1} - \sigma_1^{-1}\gamma^*\sigma_1^{-1}H_{2n}^* + \frac{d}{dt_2}[\sigma_1^{-1}]H_{2n}^*,$$

Here we can use formula (5.4) in order to differentiate the expression $H_{2n-1-j}\sigma_1^{-1}H_j^*$. From (5.4) it follows that
(5.9)

$$\begin{aligned}
&\frac{d}{dt_2}[H_k\sigma_1^{-1}H_m^*] = \\
&\quad = [\sigma_1^{-1}\sigma_2 H_{k+1} - H_{k+1}\sigma_1^{-1}\sigma_2 + \sigma_1^{-1}\gamma_* H_k - H_k\sigma_1^{-1}\gamma]\sigma_1^{-1}H_m^* + \\
&\qquad + H_k\frac{d}{dt_2}[\sigma_1^{-1}]H_m^* + \\
&\qquad\quad + H_k\sigma_1^{-1}[H_{m+1}^*\sigma_2\sigma_1^{-1} - \sigma_2\sigma_1^{-1}H_{m+1}^* + H_m^*\gamma_*^*\sigma_1^{-1} - \gamma^*\sigma_1^{-1}H_m^*] = \\
&\quad = \sigma_1^{-1}\sigma_2 H_{k+1}\sigma_1^{-1}H_m^* + H_k\sigma_1^{-1}H_{m+1}^*\sigma_2\sigma_1^{-1} - \\
&\qquad - H_{k+1}\sigma_1^{-1}\sigma_2\sigma_1^{-1}H_m^* - H_k\sigma_1^{-1}\sigma_2\sigma_1^{-1}H_{m+1}^* + \\
&\qquad\quad + \sigma_1^{-1}\gamma_* H_k\sigma_1^{-1}H_m^* + H_k\sigma_1^{-1}H_m^*\gamma_*^*\sigma_1^{-1}
\end{aligned}$$

Now we can insert the formula (5.9) into the left hand side of the previous expression and we shall obtain

$$\begin{aligned}
&\frac{d}{dt_2}[\textstyle\sum_{j=0}^{2n-1}(-1)^{j+1}H_{2n-1-j}\sigma_1^{-1}H_j^*] - \sigma_1^{-1}\gamma_* H_{2n}\sigma_1^{-1} + H_{2n}\sigma_1^{-1}\gamma\sigma_1^{-1} - \\
&\qquad - H_{2n}\frac{d}{dt_2}\sigma_1^{-1} + \sigma_1^{-1}H_{2n}^*\gamma_*^*\sigma_1^{-1} - \sigma_1^{-1}\gamma^*\sigma_1^{-1}H_{2n}^* + \frac{d}{dt_2}[\sigma_1^{-1}]H_{2n}^* = \\
&= \textstyle\sum_{j=0}^{2n-1}(-1)^{j+1}[\sigma_1^{-1}\sigma_2 H_{2n-j}\sigma_1^{-1}H_j^* + H_{2n-1-j}\sigma_1^{-1}H_{j+1}^*\sigma_2\sigma_1^{-1} - \\
&\qquad - H_{2n-j}\sigma_1^{-1}\sigma_2\sigma_1^{-1}H_j^* - H_{2n-1-j}\sigma_1^{-1}\sigma_2\sigma_1^{-1}H_{j+1}^* + \\
&\qquad\quad + \sigma_1^{-1}\gamma_* H_{2n-1-j}\sigma_1^{-1}H_j^* + H_{2n-1-j}\sigma_1^{-1}H_j^*\gamma_*^*\sigma_1^{-1}] - \\
&\quad - \sigma_1^{-1}\gamma_* H_{2n}\sigma_1^{-1} + H_{2n}\sigma_1^{-1}\gamma\sigma_1^{-1} - \\
&\qquad - H_{2n}\frac{d}{dt_2}\sigma_1^{-1} + \sigma_1^{-1}H_{2n}^*\gamma_*^*\sigma_1^{-1} - \sigma_1^{-1}\gamma^*\sigma_1^{-1}H_{2n}^* + \frac{d}{dt_2}[\sigma_1^{-1}]H_{2n}^* =
\end{aligned}$$

It is easy to recognize telescopic sums in this expression and we obtain that finally it equals to

$$\begin{aligned}
&= \textstyle\sum_{j=0}^{2n-1}(-1)^{j+1}[\sigma_1^{-1}\sigma_2 H_{2n-j}\sigma_1^{-1}H_j^* + H_{2n-1-j}\sigma_1^{-1}H_{j+1}^*\sigma_2\sigma_1^{-1}] + \\
&\quad + H_{2n}\sigma_1^{-1}\sigma_2\sigma_1^{-1}H_0^* - H_0\sigma_1^{-1}\sigma_2\sigma_1^{-1}H_{2n}^* + \\
&\qquad + \textstyle\sum_{j=0}^{2n-1}(-1)^{j+1}[\sigma_1^{-1}\gamma_* H_{2n-1-j}\sigma_1^{-1}H_j^* + H_{2n-1-j}\sigma_1^{-1}H_j^*\gamma_*^*\sigma_1^{-1}] - \\
&\quad - \sigma_1^{-1}\gamma_* H_{2n}\sigma_1^{-1} + H_{2n}\sigma_1^{-1}\gamma\sigma_1^{-1} - \\
&\qquad - H_{2n}\frac{d}{dt_2}\sigma_1^{-1} + \sigma_1^{-1}H_{2n}^*\gamma_*^*\sigma_1^{-1} - \sigma_1^{-1}\gamma^*\sigma_1^{-1}H_{2n}^* + \frac{d}{dt_2}[\sigma_1^{-1}]H_{2n}^* = \\
&= \textstyle\sum_{j=0}^{2n}(-1)^{j+1}[\sigma_1^{-1}\sigma_2 H_{2n-j}\sigma_1^{-1}H_j^* - H_{2n-j}\sigma_1^{-1}H_j^*\sigma_2\sigma_1^{-1}] + \\
&\quad + \textstyle\sum_{j=0}^{2n-1}(-1)^{j+1}[\sigma_1^{-1}\gamma_* H_{2n-1-j}\sigma_1^{-1}H_j^* + H_{2n-1-j}\sigma_1^{-1}H_j^*\gamma_*^*\sigma_1^{-1}] - \\
&\quad - \sigma_1^{-1}\gamma_* H_{2n}\sigma_1^{-1} + \sigma_1^{-1}H_{2n}^*\gamma_*^*\sigma_1^{-1} =
\end{aligned}$$

In other words, we obtain that the compatibility condition (5.7) becomes

$$\begin{aligned}
&\textstyle\sum_{j=0}^{2n-1}(-1)^{j+1}[\sigma_1^{-1}\gamma_* H_{2n-1-j}\sigma_1^{-1}H_j^* + H_{2n-1-j}\sigma_1^{-1}H_j^*\gamma_*^*\sigma_1^{-1}] - \\
&\qquad - \sigma_1^{-1}\gamma_* H_{2n}\sigma_1^{-1} + \sigma_1^{-1}H_{2n}^*\gamma_*^*\sigma_1^{-1} = 0
\end{aligned}$$

or after multiplying by σ_1 from both sides

$$\gamma_* H_{2n} - H_{2n}^* \gamma_*^* = \sum_{j=0}^{2n-1} (-1)^{j+1} [\gamma_* H_{2n-1-j} \sigma_1^{-1} H_j^* \sigma_1 + \sigma_1 H_{2n-1-j} \sigma_1^{-1} H_j^* \gamma_*^*],$$

□

Theorem 5.5. *Suppose that we are given moments $H_i(t_2)$, defined in a neighborhood of the point $t_2^0 \in \mathrm{I}$. Then there exists $n_0 \leq p^2$ such that each element H_{i+1} is uniquely determined from $H_0, \ldots, H_i$ using the algebraic formulas* (5.5)*, and n_0 LDEs with arbitrary initial conditions, obtained from* (5.4)*. Moreover, n_0 equations must be satisfied by the elements of $\gamma_*(t_2)$.*

Remarks: 1. This theorem is of local nature. The number "n_0" appearing in the theorem may vary with the point t_2^0, but is unchanged in a small neighborhood of t_2^0 by continuity.
2. The proof of the theorem allows to produce an algorithm to determine explicitly, up to n_0 initial conditions, the Markov moments. The arguments in the proof of the theorem are illustrated on an example in the following subsection. This example exhibits all the difficulties present in the general case.

Proof of 5.5: We have seen in lemma 5.4, that the real part of $H_{2n+1} = R + iM$, defined by equation (5.4) satisfies (5.7). Let as concentrate on the equation (5.8) in order to solve for M. For example, if

$$\operatorname{spec} \sigma_1^{-1} \sigma_2 \cap \operatorname{spec}(-\sigma_2 \sigma_1^{-1}) = \emptyset$$

then one can uniquely solve for M. But in general, there will be some undefined entries of M in this manner. In the later case, we can rewrite these equations as linear equations in the entries of M. But before we do that we also rewrite the equation (5.8) as following

$$\begin{aligned} 2i[\sigma_1^{-1}\sigma_2 M_{2n+1} - M_{2n+1}\sigma_2\sigma_1^{-1}] = \\ = \frac{d}{dt_2} R_{2n} - \sigma_1^{-1}\gamma_* H_{2n}\sigma_1^{-1} - H_{2n}\sigma_1^{-1}\gamma^*\sigma_1^{-1} - \\ -\sigma_1^{-1} H_{2n}^*\gamma_*^*\sigma_1^{-1} - \sigma_1^{-1}\gamma\sigma_1^{-1} H_{2n}^* \end{aligned} \tag{5.10}$$

And further, we are able to rewrite this equation as a system of p^2 linear equations in p^2 variables M^{kl}, $k, l = 1, \ldots, p^2$:

$$\begin{cases} \sum \alpha_{kl}^{11} M_{2n+1}^{kl} = \frac{d}{dt_2} R_{2n}^{11} + \sum[\beta_{kl}^{11} R_{2n}^{kl}] + \sum[\delta_{kl}^{11} M_{2n}^{kl}] \\ \sum \alpha_{kl}^{12} M_{2n+1}^{kl} = \frac{d}{dt_2} R_{2n}^{12} + \sum[\beta_{kl}^{12} R_{2n}^{kl}] + \sum[\delta_{kl}^{12} M_{2n}^{kl}] \\ \vdots \\ \sum \alpha_{kl}^{pp} M_{2n+1}^{kl} = \frac{d}{dt_2} R_{2n}^{pp} + \sum[\beta_{kl}^{pp} R_{2n}^{kl}] + \sum[\delta_{kl}^{pp} M_{2n}^{kl}] \end{cases} \tag{5.11}$$

Here $\alpha^{ij}_{kl}, \beta^{ij}_{kl}, \delta^{ij}_{kl}$ are functions of t_2 derived from the vessel parameters. Using basic algebra manipulations, it is enough to find a maximally independent subset of equations in the left side of (5.11), and to obtain a system of $p^2 - n_0$ independent equations

$$\left\{\begin{array}{l} \sum_{kl} \alpha^1_{kl}(t_2) M^{kl}_{2n+1} = f_1(\frac{d}{dt_2} R^{kl}_{2n}, R^{kl}_{2n}, M^{kl}_{2n}) \\ \sum_{kl} \alpha^2_{kl}(t_2) M^{kl}_{2n+1} = f_2(\frac{d}{dt_2} R^{kl}_{2n}, R^{kl}_{2n}, M^{kl}_{2n}) \\ \qquad\qquad \vdots \\ \sum_{kl} \alpha^{p^2-n_0}_{kl}(t_2) M^{kl}_{2n+1} = f_{p^2-n_0}(\frac{d}{dt_2} R^{kl}_{2n}, R^{kl}_{2n}, M^{kl}_{2n}) \end{array}\right. \tag{5.12}$$

for linear functions f_j and a system of linear dependent ones

$$\left\{\begin{array}{l} 0 = g_1(\frac{d}{dt_2} R_{2n}, R^{kl}_{2n}, M^{kl}_{2n}) \\ 0 = g_2(\frac{d}{dt_2} R_{2n}, R^{kl}_{2n}, M^{kl}_{2n}) \\ \vdots \\ 0 = g_{n_0}(\frac{d}{dt_2} R_{2n}, R^{kl}_{2n}, M^{kl}_{2n}) \end{array}\right.$$

for linear functions g_j.

On the other hand, using the same considerations for H_{2n+2}, we shall obtain n_0 equations of the same type, which are additional restrictions on M_{2n+1}

$$0 = g_j(\frac{d}{dt_2} M_{2n+1}, M_{2n+1}, R_{2n+1}), j = 1, \ldots, n_0. \tag{5.13}$$

Notice that we obtain algebraic (5.12) and differential (5.13) equations which are independent (since all algebraic and differentials are, and they are of different nature). So, if *all* elements of M_{2n+1} appear at these equations we shall obtain that one can uniquely solve them up to n_0 initial conditions:

$$\left\{\begin{array}{l} \sum_{kl} \alpha^i_{kl}(t_2) M^{kl}_{2n+1} = f_i(\frac{d}{dt_2} R^{kl}_{2n}, R^{kl}_{2n}, M^{kl}_{2n}) \\ \qquad\qquad\qquad\qquad\qquad i = 1, \ldots, p^2 - n_0, \\ 0 = g_i(\frac{d}{dt_2} M_{2n+1}, M_{2n+1}, R_{2n+1}), \\ \qquad\qquad\qquad\qquad\qquad i = 1, \ldots, n_0 \end{array}\right. \tag{5.14}$$

(See example of NLS at section 5.6). If this is not the case, some of the elements, say $p_0 \leq n_0$ will not appear in the equations (5.14). As a result, using the algebraic expressions (5.12) and plugging them into

differential ones, (5.13) we shall obtain $p_0 \leq n_0$ differential equation of second order on the elements R_{2n} and M_{2n}:

$$0 = h_i(\frac{d^2}{dt_2^2}R_{2n}, \frac{d}{dt_2}R_{2n}, R_{2n}, \frac{d}{dt_2}M_{2n}, M_{2n}), j = 1, \ldots, p_0.$$

By induction the same type of equation will hold for H_{2n+1} (for $j = 1, \ldots, p_0$):

$$0 = h_i(\frac{d^2}{dt_2^2}M_{2n+1}, \frac{d}{dt_2}M_{2n+1}, M_{2n+1}, \frac{d}{dt_2}R_{2n+1}, R_{2n+1}). \tag{5.15}$$

Finally, we shall obtain the following system of differential equations

$$\begin{cases} \sum_{kl} \alpha^i_{kl}(t_2)M^{kl}_{2n+1} = f_i(\frac{d}{dt_2}R^{kl}_{2n}, R^{kl}_{2n}, M^{kl}_{2n}) \\ \qquad\qquad\qquad\qquad i = 1, \ldots, p^2 - n_0, \\ 0 = g_i(\frac{d}{dt_2}M_{2n+1}, M_{2n+1}, R_{2n+1}), \\ \qquad\qquad\qquad\qquad i = 1, \ldots, n_0 - p_0 \\ 0 = h_i(\frac{d^2}{dt_2^2}M_{2n+1}, \frac{d}{dt_2}M_{2n+1}, M_{2n+1}, \frac{d}{dt_2}R_{2n+1}, R_{2n+1}), \\ \qquad\qquad\qquad\qquad i = 1, \ldots, p_0 \end{cases} \tag{5.16}$$

□

Remark: This theorem means that the equations (5.4) and (5.5) are compatible, i.e. one can always solve them, once n_0 equations are satisfied for $\gamma_*(t_2)$. But if we consider the differential equation (5.4) only and ignore (5.5), then one obtain the following

Corollary 5.6. *Suppose that Moments of H_i satisfy the differential equations* (5.4)*, then there exist $K_0(t_2), K_1(t_2), K_2(t_2), K_3(t_2)$ depending on vessel parameters only such that*

$$\frac{d}{dt_2}H_{n+1} = K_0H_n + K_1\frac{d}{dt_2}H_n + K_2\frac{d^2}{dt_2^2}H_n + K_3\frac{d^3}{dt_2^3}H_n \tag{5.17}$$

Proof: All we have to do it to imitate the proof of theorem 5.5 ignoring the additional constrain (5.5) on the real/imaginary parts of the moments. Denoting by H_n^{ij} the entries of H_n, we will obtain from (5.4)

p^2 equations (like (5.11))

$$\begin{cases} \sum \alpha_{kl}^{11} H_{n+1}^{kl} = \dfrac{d}{dt_2} H_n^{kl} + \sum [\beta_{kl}^{11} H_n^{kl}] \\ \sum \alpha_{kl}^{12} H_{n+1}^{kl} = \dfrac{d}{dt_2} H_n^{kl} + \sum [\beta_{kl}^{12} H_n^{kl}] \\ \qquad\qquad \vdots \\ \sum \alpha_{kl}^{pp} H_{n+1}^{kl} = \dfrac{d}{dt_2} H_n^{kl} + \sum [\beta_{kl}^{pp} H_n^{kl}] \end{cases}$$

and analogously to deriving the formulas (5.14), we will obtain

$$\begin{cases} \sum_{kl} \alpha_{kl}^{i}(t_2) H_{n+1}^{kl} = f_i(\dfrac{d}{dt_2} H_n, H_n), & i = 1, \ldots, p^2 - n_0, \\ 0 = g_i(\dfrac{d}{dt_2} H_{n+1}, H_{n+1}), & i = 1, \ldots, n_0 \end{cases}$$

So, if these equations are independent, differentiating the first one, we shall obtain

$$\frac{d}{dt_2} H_{n+1} = K_0 H_n + K_1 \frac{d}{dt_2} H_n + K_2 \frac{d^2}{dt_2^2} H_n.$$

Otherwise, there will be p_0 equations analogous to (5.15) of the form

$$0 = h_i(\frac{d^2}{dt_2^2} H_{n+1}, \frac{d}{dt_2} H_{n+1}, H_{n+1}), j = 1, \ldots, p_0.$$

Moreover, the independent entries of H_{n+1} will appear with one differentiation only, because, otherwise they would appear at the previous stage at the equation $0 = g_i(\frac{d}{dt_2} H_{n+1}, H_{n+1})$, and would be derived from n_0 differential equations. So, this last equation means that the remaining p_0 elements are found from

$$\frac{d}{dt_2} H_{n+1}^{ij} = L_0 H_{n+1} + L_1 \frac{d}{dt_2} H_{n+1} + L_2 \frac{d^2}{dt_2^2} H_{n+1}$$

so that only the known entries appear at the left hand side. From here it immediately follows the corollary. $\square$

We want to present next a necessary restriction on $\gamma_*(t_2)$, derived from the existence of a finite dimensional vessel:

Theorem 5.7. *Let $\sigma_1, \sigma_2, \gamma, \gamma_*$ be vessel parameters, and $\mathfrak{V}$ a finite dimensional vessel corresponding to the m parameters. Then the entries of the function γ_* satisfies a polynomial differential equation of finite order with coefficients in the differential ring $\boldsymbol{\mathcal{R}}$, generated by (the entries of) $\sigma_1, \sigma_1^{-1}, \sigma_2, \gamma$.*

Proof: Suppose that the transfer function of the vessel $\mathfrak{V}$, defined in (2.11) is

$$S(\lambda, t_2) = I - B^*(t_2)\mathbb{X}^{-1}(t_2)(\lambda I - A_1)^{-1}B(t_2)\sigma_1(t_2) = I - \sum_{i=0}^{\infty} \frac{H_i}{\lambda^{i+1}},$$

and the linkage condition is

$$\sigma_1^{-1}(t_2)\gamma_*(t_2) = \sigma_1^{-1}(t_2)\gamma(t_2) + [\sigma_1^{-1}(t_2)\sigma_2(t_2), H_0(t_2)].$$

Notice that if we differentiate this formula and use the equation for the derivative of $H_0(t_2)$ from the equation (5.4), we shall get

$$\begin{aligned} \frac{d}{dt_2}\big(\sigma_1^{-1}(t_2)\gamma_*(t_2)\big) &= \frac{d}{dt_2}\big(\sigma_1^{-1}(t_2)\gamma(t_2)\big) + [\frac{d}{dt_2}(\sigma_1^{-1}(t_2)\sigma_2(t_2)), H_0(t_2)] + \\ &+ [\sigma_1^{-1}(t_2)\sigma_2(t_2), \frac{d}{dt_2}H_0(t_2)] = \\ &= \frac{d}{dt_2}\big(\sigma_1^{-1}(t_2)\gamma(t_2)\big) + f_{00}(H_0(t_2)) + f_{01}(H_1(t_2)) \end{aligned}$$

for linear in moments functions f_{00}, f_{01} with coefficients depending on $\boldsymbol{\mathcal{R}}$ and γ_*. Similarly, differentiating this expression and using formula (5.4) for $\frac{d}{dt_2}H_0(t_2)$ and $\frac{d}{dt_2}H_1(t_2)$, we shall obtain that the second derivative is

$$\begin{aligned} &\frac{d^2}{dt_2^2}\big(\sigma_1^{-1}(t_2)\gamma_*(t_2)\big) = \\ &\quad = \frac{d^2}{dt_2^2}\big(\sigma_1^{-1}(t_2)\gamma(t_2)\big) + f_{10}(H_0) + f_{11}(H_1(t_2)) + f_{12}(H_2(t_2)). \end{aligned}$$

for linear in the moments functions f_{10}, f_{11}, f_{12} with coefficients depending on $\boldsymbol{\mathcal{R}}$ and $\gamma_*, \frac{d}{dt_2}\gamma_*$. Continuing this differentiation further at each step i we shall obtain an equation of the form

$$\frac{d^i}{dt_2^i}\big(\sigma_1^{-1}(t_2)\gamma_*(t_2)\big) = \frac{d^i}{dt_2^i}\big(\sigma_1^{-1}(t_2)\gamma(t_2)\big) + \sum_{j=0}^{i} f_{ij}(H_j),$$

where f_{ij} is a linear function of H_j with coefficients, depending on $\boldsymbol{\mathcal{R}}$ and first $i-1$ derivatives of γ_* (this can be immediately seen by the induction). But at some stage the moments start to repeat themselves

due to equation (5.1). So, taking K derivatives we shall obtain equations of the following form

(5.18)
$$\begin{cases} \sigma_1^{-1}(t_2)\gamma_*(t_2) = \sigma_1^{-1}(t_2)\gamma(t_2) + f_{00}(H_0(t_2)) \\ \dfrac{d}{dt_2}\big(\sigma_1^{-1}(t_2)\gamma_*(t_2)\big) = \dfrac{d}{dt_2}\big(\sigma_1^{-1}(t_2)\gamma(t_2)\big) + f_{10}(H_0(t_2)) + f_{11}(H_1(t_2)) \\ \dfrac{d^2}{dt_2^2}\big(\sigma_1^{-1}(t_2)\gamma_*(t_2)\big) = \dfrac{d^2}{dt_2^2}\big(\sigma_1^{-1}(t_2)\gamma(t_2)\big) + \sum\limits_{j=0}^{2} f_{2j}(H_j) \\ \vdots \\ \dfrac{d^N}{dt_2^N}\big(\sigma_1^{-1}(t_2)\gamma_*(t_2)\big) = \dfrac{d^N}{dt_2^N}\big(\sigma_1^{-1}(t_2)\gamma(t_2)\big) + \sum\limits_{j=0}^{N} f_{Nj}(H_j) \\ \vdots \\ \dfrac{d^K}{dt_2^K}\big(\sigma_1^{-1}(t_2)\gamma_*(t_2)\big) = \dfrac{d^K}{dt_2^K}\big(\sigma_1^{-1}(t_2)\gamma(t_2)\big) + \sum\limits_{j=0}^{N} f_{Kj}(H_j) \end{cases}$$

Suppose that that the dimension of the inner state is n (i.e. $\dim \mathcal{H} = n$), then we get that each of the matrices H_j has n^2 entries and there is the total number of $n^2(N+1)$ entries for the moments $H_0, \dots, H_N$. So, taking "enough" derivatives of $\gamma_*(t_2)$ (i.e., taking K so that $K\dim(\mathcal{E})^2 > n^2(N+1)$) and eliminating all the entries of the moments, we shall obtain a finite number of polynomial differential equation for the entries of $\gamma_*(t_2)$. $\square$

Remark: From this theorem is follows that γ_* satisfies an equation of the form
$$P(x, x', x'', \dots, x^{(K)}) = 0,$$
where $P(x_0, x_1, x_2, \dots, x_K)$ is a non-commutative polynomial with coefficients in $\boldsymbol{\mathcal{R}}$.

Using an outer space transformation it is possible to bring the matrix $\sigma_1^{-1}(t_2)\sigma_2(t_2)$ to a simpler form. Suppose that there is an invertible matrix $V(t_2): \mathcal{E} \to \mathcal{E}$ such that
$$\sigma_1^{-1}(t_2)\sigma_2(t_2) = V(t_2)L(t_2)V^{-1}(t_2)$$
for a "simple" matrix $L(t_2)$, where $L(t_2^0)$ is in a Jordan block form. Define new operators (omitting t_2-dependence)

(5.19)
$$\widetilde{\sigma}_1 = V^*\sigma_1 V, \quad \widetilde{\sigma}_2 = V^*\sigma_2 V, \quad \widetilde{\gamma} = V^*\gamma V - \sigma_1 V', \quad \widetilde{\gamma}_* = V^*\gamma_* V - \sigma_1 V'$$

then simple calculations show that $\widetilde{\sigma}_1(t_2), \widetilde{\sigma}_2(t_2), \widetilde{\gamma}(t_2), \widetilde{\gamma}_*(t_2)$ are vessel parameters in the same interval I and the collection of operators and spaces
$$\widetilde{\mathfrak{V}} = (A_1, B(t_2), \mathbb{X}(t_2); \widetilde{\sigma}_1(t_2), \widetilde{\sigma}_2(t_2), \widetilde{\gamma}(t_2), \widetilde{\gamma}_*(t_2); \mathcal{H}, \mathcal{E}),$$

is again a Vessel. By multiplying the original vessel conditions of $\mathfrak{V}$ by V or on V^* we check that the the vessel conditions are satisfied for $\widetilde{\mathfrak{V}}$. But for this new vessel $\widetilde{\mathfrak{V}}$ we obtain that

$$\widetilde{\sigma}_1^{-1}(t_2)\widetilde{\sigma}_2(t_2) = V^{-1}(t_2)\sigma_1^{-1}(t_2)\sigma_2(t_2)V(t_2) = L(t_2).$$

The first step to understand the equations arising in theorem (5.5) we study the cosntant case.

5.3. **Moment equations for diagonal constant $\sigma_1^{-1}\sigma_2$.** At the special case $\sigma_1^{-1}\sigma_2 = \operatorname{diag}[s_1, s_2, \ldots, s_E]$ ($E = \dim \mathcal{E}$) for some constants $s_i \in \mathbb{C}$ we can explicitly find the formulas for the moments using equation (5.4).

First we consider the linkage condition (1.3), which may be considered as the "-1" equation of (5.4). Taking diagonal form for $\sigma_1^{-1}(t_2)\sigma_2(t_2)$ and denoting the k,j-th entry of $H_0(t_2)$ by H_0^{kj}, we shall obtain

$$\sigma_1^{-1}(t_2)[\gamma_*(t_2) - \gamma(t_2)] = [\sigma_1^{-1}(t_2)\sigma_2(t_2), H_0(t_2)] = \left[\ H_0^{kj}(t_2)(s_k - s_j)\ \right].$$

Cases for which $s_k - s_j = 0$ give restrictions on the entries of $\sigma_1^{-1}(t_2)[\gamma_*(t_2) - \gamma(t_2)]$ and other cases uniquely determine entries of $H_0(t_2)$:

$$e_k\sigma_1^{-1}(t_2)[\gamma_*(t_2) - \gamma(t_2)]e_j^t = 0, \quad s_k - s_j = 0, \tag{5.20}$$

$$H_0^{kj}(t_2) = \frac{e_k\sigma_1^{-1}(t_2)[\gamma_*(t_2) - \gamma(t_2)]e_j^t}{s_k - s_j}, \quad s_k - s_j \neq 0. \tag{5.21}$$

But there more equations arise when we consider (5.4) for $i = 0$:

$$\begin{aligned}&[\sigma_1^{-1}\sigma_2, H_1(t_2)] = \left[\ H_1^{kj}(t_2)(s_k - s_j)\ \right] = \\ &= \frac{d}{dt_2}H_0(t_2) - \sigma_1^{-1}(t_2)\gamma_*(t_2)H_0(t_2) + H_0(t_2)\sigma_1^{-1}(t_2)\gamma(t_2)\end{aligned}$$

Here again if we focus on the cases $s_k - s_j = 0$, we shall obtain that the following differential equations must hold (suppressing notation for t_2 dependence)

$$\frac{d}{dt_2}H_0^{kj} - e_k[\sigma_1^{-1}\gamma_*H_0 + H_0\sigma_1^{-1}\gamma]e_j^t = 0, \quad s_k - s_j = 0,$$

$$H_1^{kj}(t_2) = \frac{1}{s_k - s_j}e_k[\frac{d}{dt_2}H_0 - \sigma_1^{-1}\gamma_*H_0 + H_0\sigma_1^{-1}\gamma]e_j^t, \quad s_k - s_j \neq 0$$

The first line gives a linear differential equation for the elements H_0^{kj} in terms of vessel parameters and other known entries of H_0:

$$\frac{d}{dt_2}H_0^{kj} = e_k\sigma_1^{-1}\gamma_*H_0e_j^t - e_kH_0\sigma_1^{-1}\gamma e_j^t, \quad 1 \leq k \leq r,$$

Finally we obtain the following

Lemma 5.8. *Suppose that*

$$\sigma_1^{-1}\sigma_2 = \operatorname{diag}[s_1, s_2, \ldots, s_E],$$

then the entries of γ, γ_* *satisfy the following equations*

$$e_k \sigma_1^{-1}(t_2)[\gamma_*(t_2) - \gamma(t_2)]e_k^t = 0, \qquad \textit{when } s_k - s_j = 0, \tag{5.22}$$

where e_k *stay for the standard elementary row vector with 1 at k-th place and* 0 *everywhere.*

An interesting question arises, is whether any continuous vessel parameters satisfying the conditions of this lemma gives rise to a transfer function.

5.4. **Moment equations for Jordan-block constant $\sigma_1^{-1}\sigma_2$.** Suppose now that $\sigma_1^{-1}\sigma_2$ is a Jordan block matrix with arbitrary eigenvalue $s \in \mathbb{C}$.

$$\sigma_1^{-1}\sigma_2 = \operatorname{Jordan}(s, r) = \begin{bmatrix} s & 1 & \ldots & 0 & 0 \\ 0 & s & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & s & 1 \\ 0 & 0 & \ldots & 0 & s \end{bmatrix}$$

then we can again consider the linkage condition (1.3):

$$\begin{aligned}
&\sigma_1^{-1}(t_2)[\gamma_*(t_2) - \gamma(t_2)] = [\sigma_1^{-1}(t_2)\sigma_2(t_2), H_0(t_2)] = \\
&= [\operatorname{Jordan}(s, r), H_0(t_2)] = [\operatorname{Jordan}(0, r), H_0(t_2)] = \\
&= \begin{bmatrix}
H_0^{21} & H_0^{22} - H_0^{11} & H_0^{23} - H_0^{12} & \ldots & H_0^{2r} - H_0^{1,r-1} \\
H_0^{31} & H_0^{32} - H_0^{21} & H_0^{33} - H_0^{22} & \ldots & H_0^{3r} - H_0^{2,r-1} \\
H_0^{41} & H_0^{42} - H_0^{31} & H_0^{43} - H_0^{32} & \ldots & H_0^{4r} - H_0^{3,r-1} \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
H_0^{r-1,1} & H_0^{r-1,2} - H_0^{r-2,1} & H_0^{r-1,3} - H_0^{r-2,2} & \ldots & H_0^{r-1,r} - H_0^{r-2,r-1} \\
H_0^{r1} & H_0^{r2} - H_0^{r-1,1} & H_0^{r3} - H_0^{r-1,2} & \ldots & H_0^{rr} - H_0^{r-1,r-1} \\
0 & -H_0^{r1} & -H_0^{r2} & \ldots & -H_0^{r,r-1}
\end{bmatrix},
\end{aligned}$$

where the value s on the main diagonal is canceled. It turns out that sums of elements of each diagonal under the main one is 0. This may be rewritten also in the following form

$$\operatorname{tr}[\sigma_1^{-1}(\gamma_*(t_2) - \gamma(t_2)) \operatorname{Jordan}(0, r)^k] = 0, \quad k = 0, 1, \ldots, r-1.$$

From the other equations we find expressions for the entries of $H_0(t_2)$ using its first row as parameters. In other words, from the linkage condition (1.3) we find r restrictions on $\sigma_1^{-1}(\gamma_*(t_2) - \gamma(t_2))$ and all the entries of $H_0(t_2)$, while the entries of the first row $H_0^{11}, \ldots, H_0^{1r}$ are unknown.

Considering next the equation (5.4), we will similarly obtain that

$$\operatorname{tr}[\big(\frac{d}{dt_2}H_0 - \sigma_1^{-1}\gamma_* H_0 + H_0\sigma_1^{-1}\gamma\big)\ \mathrm{Jordan}(0,r)^k] = 0, k = 0, 1, \ldots, r-1.$$

These equations may be considered as additional equations on the first row of $H_0(t_2)$, which was unknown before. It is also possible that these differential equations will not determine the first row of H_0, in this case we shall obtain additional differential restrictions on the elements of $\sigma_1^{-1}(\gamma_*(t_2) - \gamma(t_2))$ (see (5.24) below). In this case, the elements of the first row of H_0 will be determined from differential equations on the next level, involving $H_1(t_2)$ (see (5.32) below).

5.5. **Sturm Liouville vessel parameters.** The following example was extensively studied in [M2]. It deals with the Sturm Liouville differential equation

$$\frac{d^2}{dt_2^2}y(t_2) - q(t_2)y(t_2) = \lambda y(t_2),$$

with the spectral parameter λ. The parameter $q(t_2)$ is usually called the *potential*. For $q(t_2) = 0$ this problem is easily solved by exponents and in this case we shall call this equation *trivial*. In [M2] one connects solutions of the more general problem with non trivial $q(t_2)$ to the trivial one.

Definition 5.9. *the Sturm Liouville vessel parameters are given by*

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \sigma_2 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \gamma = \begin{bmatrix} 0 & 0 \\ 0 & i \end{bmatrix},$$
$$\gamma_*(t_2) = \begin{bmatrix} -i\pi_{11}(t_2) & -\beta(t_2) \\ \beta(t_2) & i \end{bmatrix}$$

for ***real*** *valued continuous functions* $\pi_{11}(t_2), \beta(t_2)$.

Notice that these parameters correspond to the Jordan block form of $\sigma_1^{-1}\sigma_2 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$.

The input compatibility differential equation (2.16) is equivalent to

$$\begin{cases} \lambda u_1(\lambda, t_2) - \frac{\partial}{\partial t_2}u_2(\lambda, t_2) = 0 \\ -\frac{\partial}{\partial t_2}u_1(\lambda, t_2) + iu_2(\lambda, t_2) = 0 \end{cases}$$

where we denote $u_\lambda(t_2) = \begin{bmatrix} u_1(\lambda, t_2) \\ u_2(\lambda, t_2) \end{bmatrix}$. From the second equation one finds that $u_2(\lambda, t_2) = -i\frac{\partial}{\partial t_2}u_1(\lambda, t_2)$ and plugging it back to the

first equation, we shall obtain the trivial Sturm Liouville differential equation with the spectral parameter $i\lambda$ for $u_1(\lambda, t_2)$:

$$\frac{\partial^2}{\partial t_2^2} u_1(\lambda, t_2) = i\lambda u_1(\lambda, t_2).$$

For the output $y_\lambda(t_2) = \begin{bmatrix} y_1(\lambda, t_2) \\ y_2(\lambda, t_2) \end{bmatrix}$, we shall obtain that (2.17) is equivalent to the system of equations

$$\begin{cases} (\lambda - i\pi_{11}(t_2))y_1(\lambda, t_2) - (\frac{\partial}{\partial t_2} + \beta(t_2))y_2(\lambda, t_2) = 0 \\ (\beta(t_2) - \frac{\partial}{\partial t_2})y_1(\lambda, t_2) + iy_2(\lambda, t_2) = 0 \end{cases},$$

from which we immediately obtain that $y_2(\lambda, t_2) = i(\beta(t_2) - \frac{\partial}{\partial t_2})y_1(\lambda, t_2)$ and plugging it into the first equation

$$\frac{\partial^2}{\partial t_2^2} y_1(\lambda, t_2) - (\pi_{11}(t_2) + \beta'(t_2) + \beta^2(t_2))y_1(\lambda, t_2) = i\lambda y_1(\lambda, t_2),$$

which means that $y_1(\lambda, t_2)$ satisfies the Sturm Liouville differential equation with the spectral parameter $i\lambda$ and the potential $q(t_2) = (\pi_{11}(t_2) + \beta'(t_2) + \beta^2(t_2))$.

The first equation (1.3) considered for $H_0 = \begin{bmatrix} H_0^{11} & H_0^{12} \\ H_0^{21} & H_0^{22} \end{bmatrix}$ becomes

$$\begin{bmatrix} -i\pi_{11}(t_2) & -\beta(t_2) \\ \beta(t_2) & 0 \end{bmatrix} = \begin{bmatrix} H_0^{11} - H_0^{22} & H_0^{12} \\ -H_0^{12} & 0 \end{bmatrix}.$$

from where we conclude that

$$H_0^{12} = -\beta(t_2), \quad H_0^{11} - H_0^{22} = -i\pi_{11}(t_2). \tag{5.23}$$

Let us consider the differential equation (5.4), where we use the notation $H_1 = \begin{bmatrix} H_1^{11} & H_1^{12} \\ H_1^{21} & H_1^{22} \end{bmatrix}$. Substituting the expressions for the vessel parameters, we shall obtain

$$\begin{cases} \frac{d}{dt_2} H_0^{11} - \beta H_0^{11} - iH_0^{21} & = -H_1^{12}, \\ \frac{d}{dt_2} H_0^{12} - \beta H_0^{12} + i(H_0^{11} - H_0^{22}) & = 0, \\ \frac{d}{dt_2} H_0^{21} + i\pi_{11} H_0^{11} + \beta H_0^{21} & = H_1^{11} - H_1^{22}, \\ \frac{d}{dt_2} H_0^{22} + i\pi_{11} H_0^{12} + \beta H_0^{22} + iH_0^{21} & = H_1^{12}. \end{cases}$$

and consequently, using the formulas (5.23) the second equation results in

$$\pi_{11} = \frac{d}{dt_2}\beta - \beta^2. \tag{5.24}$$

Together the first and the fourth equations give

$$H_1^{12} = -(\frac{d}{dt_2}H_0^{11} - \beta H_0^{11} - iH_0^{21}) = \frac{d}{dt_2}H_0^{22} + i\pi_{11}H_0^{12} + \beta H_0^{22} + iH_0^{21}$$

from where we obtain using (5.23)

$$\frac{d}{dt_2}[H_0^{11} + H_0^{22}] = 0 \Rightarrow H_0^{11} + H_0^{22} = C \in \mathbb{C}, \tag{5.25}$$

Additionally, H_0 has to satisfy $H_0 = \sigma_1^{-1}H_0^*\sigma_1$. Using this relation we shall obtain that

$$\begin{bmatrix} H_0^{11} & H_0^{12} \\ H_0^{21} & H_0^{22} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} (H_0^{11})^* & (H_0^{21})^* \\ (H_0^{12})^* & (H_0^{22})^* \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \Rightarrow$$
$$\Rightarrow \begin{cases} H_0^{11} = (H_0^{22})^* \\ H_0^{12} = (H_0^{12})^* \\ H_0^{21} = (H_0^{21})^* \\ H_0^{22} = (H_0^{11})^* \end{cases}$$

from where we conclude that

$$H_0^{11} = (H_0^{22})^*, \quad H_0^{12} = (H_0^{12})^*, \quad H_0^{21} = (H_0^{21})^*. \tag{5.26}$$

As we can see $H_0^{21} = h_0^{21}$ is a real valued (arbitrary at this stage) function, and H_0 is as follows

$$H_0 = \begin{bmatrix} \dfrac{r - i\pi_{11}}{2} & -\beta \\ h_0^{21} & \dfrac{r + i\pi_{11}}{2} \end{bmatrix}, \tag{5.27}$$

where $r \in \mathbb{R}$.

Let us perform the same calculations for H_1. From the algebraic equation (5.5)

$$H_1\sigma_1^{-1} + \sigma_1^{-1}H_1^* = -H_0\sigma_1^{-1}H_0^*$$

we obtain that

$$\begin{bmatrix} H_1^{12} + (H_1^{12})^* & H_1^{11} + (H_1^{22})^* \\ H_1^{22} + (H_1^{11})^* & H_1^{21} + (H_1^{21})^* \end{bmatrix} = -H_0\sigma_1^{-1}H_0^*$$

From the equation (5.4) with $i = 0$ we obtain as before that

$$H_1^{11} - H_1^{22} = \frac{d}{dt_2}H_0^{21} + i\pi_{11}H_0^{11} + \beta H_0^{21}, \tag{5.28}$$

$$H_1^{12} = iH_0^{21} - \frac{d}{dt_2}H_0^{11} + \beta H_0^{11}. \tag{5.29}$$

and the same equation (5.4) with $i = 1$ produces similarly to the previous case

$$\frac{d}{dt_2}H_1^{12} - \beta H_1^{12} + i(H_1^{11} - H_1^{22}) = 0, \tag{5.30}$$

$$\frac{d}{dt_2}(H_1^{11} + H_1^{22}) = -i\pi_{11}H_1^{12} + \beta(H_1^{11} - H_1^{22}). \tag{5.31}$$

Plugging (5.28) and (5.29) into (5.30), we shall obtain that $H_0^{21} = h_0^{21}$ have to satisfy the following differential equation of the first order:

$$\begin{aligned} &\frac{d}{dt_2}(iH_0^{21} - \tfrac{d}{dt_2}H_0^{11} + \beta H_0^{11}) - \\ &-\beta(iH_0^{21} - \tfrac{d}{dt_2}H_0^{11} + \beta H_0^{11}) + \\ &+i\tfrac{d}{dt_2}H_0^{21} - \pi_{11}H_0^{11} + i\beta H_0^{21} = 0. \end{aligned}$$

or after cancellations

$$2i(H_0^{21})' = \frac{d^2}{dt_2^2}H_0^{11} - 2\beta\frac{d}{dt_2}H_0^{11}. \tag{5.32}$$

Inserting here the expressions for H_0^{ij} appearing in (5.27) we shall obtain that the real part of the last equality can be derived from the equation (5.24):

$$\frac{r}{2}[\beta' - \beta^2 - \pi_{11}] = 0.$$

The imaginary part gives the following equation

$$4\frac{d}{dt_2}h_0^{21} = \frac{d}{dt_2}(\pi_{11}\beta) + \beta\pi_{11}' - \beta^2\pi_{11} - \pi_{11}^2 - \pi_{11}''.$$

Suppose (see [M2]) that there exists a function τ such that $\beta = -\dfrac{\tau'}{\tau}$. Then using (5.24) $\pi_{11} = -\dfrac{\tau''}{\tau}$ and inserting these equations into the formula for $\frac{d}{dt_2}h_0^{21}$ we obtain that

$$4\frac{d}{dt_2}h_0^{21} = \frac{\tau^{(4)}}{\tau} - \Big(\frac{\tau''}{\tau}\Big)^2.$$

Let us write down the formulas for the elements of H_1:

$$\begin{cases} H_1^{12} = \frac{d}{dt_2}H_0^{21} - \dfrac{d}{dt_2}H_0^{11} + \beta H_0^{11}, & (5.29) \\ H_1^{11} - H_1^{22} = i(\frac{d}{dt_2}H_1^{12} - \beta H_1^{12}), & (5.28) \\ \frac{d}{dt_2}(H_1^{11} + H_1^{22}) = -i\pi_{11}H_1^{12} + \beta(H_1^{11} - H_1^{22}), & (5.31) \\ 2i(H_1^{21})' = \frac{d^2}{dt_2^2}H_1^{11} - 2\beta\frac{d}{dt_2}H_1^{11} & (5.32)' \end{cases}$$

The last equation $(5.32)'$ is obtained from (5.32) by substituting the index 0 at H_0^{ij} by the index 1: H_1^{ij}.
Finally, we obtain that in the general case H_{i+1} is derived from H_i using a system of similar equations.

$$(5.33) \qquad \begin{cases} H_{i+1}^{12} & = \frac{d}{dt_2}H_i^{21} - \dfrac{d}{dt_2}H_i^{11} + \beta H_i^{11}, \\ H_{i+1}^{11} - H_{i+1}^{22} & = i(\frac{d}{dt_2}H_{i+1}^{12} - \beta H_{i+1}^{12}), \\ \frac{d}{dt_2}(H_{i+1}^{11} + H_{i+1}^{22}) & = -i\pi_{11}H_{i+1}^{12} + \beta(H_{i+1}^{11} - H_{i+1}^{22}), \\ 2i\frac{d}{dt_2}H_{i+1}^{21} & = \frac{d^2}{dt_2^2}H_{i+1}^{11} - 2\beta\frac{d}{dt_2}H_{i+1}^{11}. \end{cases}$$

from where we see that Markov moments are defined up to initial conditions for $n_0 = 2$ elements. Notice that $p_0 = 1$ in this case.
Let us also demonstrate theorem 5.7 for the Sturm Liouville parameters from definition 5.9. We will take the simplest case $n = 1$ and as a result the transfer function is

$$S(\lambda, t_2) = I - \frac{1}{\lambda + z}C(t_2)B(t_2)\sigma_1 = I - \sum_{i=0}^{\infty}\frac{1}{\lambda^{i+1}}(-z)^iC(t_2)B(t_2)\sigma_1.$$

We have already seen (in (5.24)) that γ_* is necessarily of the form

$$\gamma_*(t_2) = \begin{bmatrix} -i(\beta' - \beta^2) & -\beta \\ \beta & i \end{bmatrix}$$

for a real valued function $\beta(t_2)$ on I. In this case, $N = 0$ which means that the first moment is a multiple of the zero moment: $H_1 = -zH_0$. Since the vessel parameters are constant the differential ring $\boldsymbol{\mathcal{R}} = \mathbb{C}$ is trivial. Using the formulas developed in [M2] we shall obtain that $\tau = \exp\int\beta$ satisfies

$$(\frac{d}{dt_2} - k)(\frac{d}{dt_2} - \bar{k})(\frac{d}{dt_2} + k)(\frac{d}{dt_2} + \bar{k})\tau = 0.$$

for $k = \sqrt{-iz}$, which may be rewritten as a polynomial differential equation for β, after inserting the formula for $\tau = \exp\int\beta$ and multiplying by τ^{-1}.

5.6. **Non-Linear Shrödinger equation parameters.**

Definition 5.10. *Non-Linear Shrödinger equation parameters are given by*

$$\sigma_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \sigma_2 = \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \gamma(x) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$
$$\gamma_*(x) = \begin{bmatrix} 0 & \beta(x) \\ -\beta^*(x) & 0 \end{bmatrix}$$

Then the output and output compatibility conditions take the form of the classical non linear Schrödinger equation with the spectral parameter $i\lambda$

$$\frac{\partial}{\partial x}u(x,\lambda) = (i\lambda A + Q(x))u(x,\lambda),$$

where

$$I = \sigma_1, \quad A = -\frac{1}{2}\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} = -i\sigma_2, \quad Q(x) = -\gamma_*(x).$$

We may perform the same calculations for the moments as in the Sturm-Liouville case. Using formula (1.2) and denoting n-th moment by $H_n(t_2) = \begin{bmatrix} H_n^{11} & H_n^{12} \\ H_n^{21} & H_n^{22} \end{bmatrix}$ we shall find that the moments has to satisfy the following relations

$$\begin{cases} H_n^{21} = -\frac{d}{dt_2}\big(H_{n-1}^{21}\big) - \beta^*(t_2)H_{n-1}^{11}, \\ H_n^{12} = \frac{d}{dt_2}\big(H_{n-1}^{12}\big) - \beta(t_2)H_{n-1}^{22}, \\ \frac{d}{dt_2}\big(H_n^{11}\big) = \beta(t_2)H_n^{21}, \\ \frac{d}{dt_2}\big(H_n^{22}\big) = -\beta^*(t_2)H_n^{12}. \end{cases} \tag{5.34}$$

while the first moment H_0 is found from the linkage condition (2.15)

$$H_0 = \begin{bmatrix} H_0^{11} & \beta(t_2) \\ \beta^*(t_2) & H_0^{22} \end{bmatrix},$$

and the entries H_0^{11}, H_0^{22} are found using two last equations of (5.34):

$$\frac{d}{dt_2}\big(H_0^{11}\big) = \beta(t_2)H_0^{21} = \beta(t_2)\beta^*(t_2),$$
$$\frac{d}{dt_2}\big(H_0^{22}\big) = -\beta^*(t_2)H_0^{12} = -\beta^*(t_2)\beta(t_2).$$

6. Inverse problem at infinity for given vessel parameters

In order to discuss the reconstruction of the transfer function from its moments, we will show first that for almost arbitrary vessel parameters one can solve the n_0 moment equations, appearing in Theorem 5.5. Once a symmetric, equal to identity at infinity, function is constructed, we will use a Krein space realization for it. Finally, by a counterexample, we will show that not all vessels parameters admit Hilbert space (with positive $\mathbb{X}$) realizations.

We have seen that there is a differential equation (5.17), which connects the moments, and means that the next moments is roughly speaking a second derivative of the previous one. If we start to estimate its growth, we will end with factorial growth for the moments, because consecutive applying of derivative produces many terms due to Leibniz rule. Instead, we will assume that there is a "big" function $M(t_2)$ which majorizes all the moments. Of course such an assumption narrows the class of function, to which the realization theorem 6.1 is applicable. On the other hand, the algebraic equations (5.5) connects the real/ imaginary parts of the moments with all the previous ones. Again, we found assumptions, which will enable us to handle the growth of such elements.

From theorem 5.2 it follows that we have to demand the symmetry condition (5.5) only at t_2^0. Examining the equation (5.5) and supposing that $\|H_n(t_2^0)\| \leq f_n C^{n+1}$, we will obtain that

$$\|H_{n+1}\sigma_1^{-1} + (-1)^i \sigma_1^{-1} H_{n+1}^*\| \leq \sum_{j=0}^{n} \|H_{n-j}\| \|\sigma_1^{-1}\| \|H_j^*\| \leq$$

$$\leq \|\sigma_1^{-1}\| \sum_{j=0}^{n} f_{n-j} C^{n-j+1} f_j C^{j+1} = \|\sigma_1^{-1}\| \sum_{j=0}^{n} f_{n-j} f_j C^{n+2}$$

So, if the series $\{f_n\}$ has the property

$$f_{n+1} = \|\sigma_1^{-1}\| \sum_{j=0}^{n} f_{n-j} f_j, \tag{6.1}$$

we will obtain that

$$\|H_{n+1}\sigma_1^{-1} + (-1)^i \sigma_1^{-1} H_{n+1}^*\| \leq \|\sigma_1^{-1}\| \sum_{j=0}^{n} f_{n-j} f_j C^{n+2} \leq f_{n+1} C^{n+2}.$$

One can find generating function $F(x) = \sum f_i x^i$. Multiplying (6.1) by x^{n+1} and summing, we will obtain that

$$xF(x)^2 = F(x) - F(0) \Rightarrow F(x) = \frac{1 - \sqrt{1 - 4xF(0)}}{2x} = \sum_{n=0}^{\infty} f_n x^n$$

from where it follows that for $F(0) = 1$

$$f_n \sim \frac{4^n}{\sqrt{\pi n}(2n-1)}$$

Let $M(t_2)$ be a positive function with positive derivatives on $[t_2^0, t_2]$, which satisfies the following differential equation

$$\begin{aligned}(6.2)\quad & \frac{C}{3\sqrt{2}}\frac{d}{dt_2}M(t_2) = \\ & = \|K_0\|M(t_2) + \|K_1\|\frac{d}{dt_2}M(t_2) + \|K_2\|\frac{d^2}{dt_2^2}M(t_2) + \|K_3\|\frac{d^3}{dt_2^3}M(t_2)\end{aligned}$$

then, if $H_n^{(k)}(t_2) \leq f_n C^{n+1} M^{(k)}(t_2)$ we will obtain that

$$\begin{aligned}& \|\frac{d}{dt_2}H_{n+1}\| \leq \\ & \leq f_n C^{n+1}(\|K_0\|\|H_n\|+\|K_1\|\|\frac{d}{dt_2}H_n\|+\|K_2\|\|\frac{d^2}{dt_2^2}H_n\|+\|K_3\|\|\frac{d^3}{dt_2^3}H_n\|) \\ & \leq f_n C^{n+1}(\|K_0\|M(t_2)+\|K_1\|\frac{d}{dt_2}M(t_2)+\|K_2\|\frac{d^2}{dt_2^2}M(t_2)+\|K_3\|\frac{d^3}{dt_2^3}M(t_2)) \\ & \qquad = \frac{f_n}{3\sqrt{2}}C^{n+2}M'(t_2) \leq f_{n+1}C^{n+2}M'(t_2)\end{aligned}$$

and by induction on k, further differentiating the equation (5.17) and using triangle inequality and additional assumption $\|K_i^{(k)}(t_2)\| \leq \frac{d^k}{dt_2^k}\|K_i(t_2)\|$, we will obtain that

$$\|H_{n+1}^{(k)}(t_2)\| \leq \frac{f_n}{3\sqrt{2}}C^{n+2}M^{(k)}(t_2) \leq f_{n+1}C^{n+2}M^{(k)}(t_2).$$

Using these ideas we obtain the following

Theorem 6.1. *Suppose that the vessels parameters and the norm are chosen so that K_0, K_1, K_2, K_3 satisfy for all $k \geq 1$*

$$(6.3)\qquad \|K_i^{(k)}(t_2)\| \leq \frac{d^k}{dt_2^k}\|K_i(t_2)\|, \quad i = 0, 1, 2, 3.$$

Suppose also that $M(t_2)$ is a solution of the differential equation (6.2) *with the initial value I, which majorizes $H_0(t_2)$ and all its derivatives: $\|H_0^{(k)}(t_2)\| \leq f_0 C M(t_2)$ on $[t_2^0, t_2]$. Then if at each step the imaginary (for even n) and real (for odd n) part of $H_{n+1}\sigma_1^{-1}$, which is uniquely determined from the first line in* (5.16) *also satisfies*

$$(6.4)\qquad \|H_{n+1}(t_2^0)\sigma_1^{-1} - (-1)^i\sigma_1^{-1}H_{n+1}^*(t_2^0)\| \leq f_{n+1}C^{n+2},$$

then the series

$$I - \sum_{n=0}^{\infty}\frac{1}{\lambda^{i+1}}H_n(t_2)\sigma_1$$

determines an analytic at $\lambda = \infty$ function $S(\lambda, t_2)$ not necessary $\sigma_1(t_2)$-contractive function (i.e. condition (2.19) *of proposition 2.5 does not necessary hold), which maps solutions of* (2.16) *to solutions of* (2.17) *and which is symmetric (i.e. satisfies* (2.23)*). Radius of convergence around $\lambda = \infty$ is at least $\dfrac{1}{4C}$.*

Proof: Let us write down all the ideas, preceding the claim of this theorem. We will prove the theorem, using induction on n. The basis of the induction follows from the assumptions, namely.

$$\|H_0^{(k)}\| \le f_0 C M(t_2).$$

Suppose that $\|H_i^{(k)}(t_2)\| \le f_i C^{i+1} M^{(k)}(t_2)$ for $i = 0, 1, \ldots, n$, then from (5.17) it follows that

$$\begin{aligned}
&\|\frac{d}{dt_2} H_{n+1}\| \le \\
&\le f_n C^{n+1}(\|K_0\|\|H_n\| + \|K_1\|\|\frac{d}{dt_2} H_n\| + \|K_2\|\|\frac{d^2}{dt_2^2} H_n\| + \|K_3\|\|\frac{d^3}{dt_2^3} H_n\|) \\
&\le f_n C^{n+1}(\|K_0\| M(t_2) + \|K_1\|\frac{d}{dt_2} M(t_2) + \|K_2\|\frac{d^2}{dt_2^2} M(t_2) + \|K_3\|\frac{d^3}{dt_2^3} M(t_2)) \\
&\qquad\qquad = \frac{f_n}{3\sqrt{2}} C^{n+2} M'(t_2) \le f_{n+1} C^{n+2} M'(t_2)
\end{aligned}$$

and by induction on k, we obtain that for $k \ge 1$,

$$\begin{aligned}
&\|H_{n+1}^{(k)}(t_2)\| = \\
&= \|\sum_{i=0}^{k} \binom{k}{i} \{K_0^{(i)} H_n^{(k-i-1)} + K_1^{(i)} H_n^{(k-i)} + K_2^{(i)} H_n^{(k-i+1)} + K_3^{(i)} H_n^{(k-i+2)}\} \\
&\le \sum_{i=0}^{k} \binom{k}{i} \{\|K_0^{(i)}\|\|H_n^{(k-i-1)}\| + \|K_1^{(i)}\|\|H_n^{(k-i)}\| + \|K_2^{(i)}\|\|H_n^{(k-i+1)}\| + \|K_3^{(i)}\|\|H_n^{(k-i+2)}\|\}
\end{aligned}$$

using assumption 6.3

$$\begin{aligned}
&\le \sum_{i=0}^{k} \binom{k}{i} \{\|K_0\|^{(i)}\|H_n^{(k-i-1)}\| + \|K_1\|^{(i)}\|H_n^{(k-i)}\| + \|K_2\|^{(i)}\|H_n^{(k-i+1)}\| + \|K_3\|^{(i)}\|H_n^{(k-i+2)}\|\} \\
&\le f_n C^{n+1} \sum_{i=0}^{k} \binom{k}{i} \{\|K_0\|^{(i)} M^{(k-i-1)} + \|K_1\|^{(i)} M^{(k-i)} + \|K_2\|^{(i)} M^{(k-i+1)} + \|K_3\|^{(i)} M^{(k-i+2)} = \\
&\qquad\qquad = f_n C^{n+1} (\frac{C}{3\sqrt{2}} M')^{(k-1)} \le f_{n+1} C^{n+2} M^{(k)}(t_2).
\end{aligned}$$

$$\|H_{n+1}(t_2^0)\sigma_1^{-1}+(-1)^i\sigma_1^{-1}H_{n+1}^*(t_2^0)\| \le \sum_{j=0}^{n}\|H_{n-j}(t_2^0)\|\|\sigma_1^{-1}\|\|H_j^*(t_2^0)\| \le$$
$$\le \|\sigma_1^{-1}\|\sum_{j=0}^{n} f_{n-j}C^{n-j+1}f_jC^{j+1} = \|\sigma_1^{-1}\|\sum_{j=0}^{n} f_{n-j}f_jC^{n+2} \le f_{n+1}C^{n+2}$$

From (6.4) and this inequality it follows that

$$\|H_{n+1}(t_2^0)\sigma_1^{-1}\| \le \frac{1}{2}\|H_{n+1}(t_2^0)\sigma_1^{-1}+(-1)^i\sigma_1^{-1}H_{n+1}^*(t_2^0)\|+$$
$$+\frac{1}{2}\|H_{n+1}(t_2^0)\sigma_1^{-1}-(-1)^i\sigma_1^{-1}H_{n+1}^*(t_2^0)\| \le$$
$$\le \frac{1}{2}(f_{n+1}C^{n+2}+f_{n+1}C^{n+2}) = f_{n+1}C^{n+2}.$$

Then

$$\|H_{n+1}(t_2)\sigma_1^{-1}\| = \|H_{n+1}(t_2^0)\sigma_1^{-1}+\int_{t_2^0}^{t_2}\|H'_{n+1}(y)\|dy \le$$
$$\le f_{n+1}C^{n+2}+\int_{t_2^0}^{t_2} f_{n+1}C^{n+2}M'(y)dy =$$
$$= f_{n+1}C^{n+2}+f_{n+1}C^{n+2}(M(t_2)-I) = f_{n+1}C^{n+2}M(t_2).$$

From these estimates we obtain that

$$H_n(t_2) \le f_nC^{n+1}M(t_2)$$

and as a result its radius of convergence is

$$\frac{1}{R} = \lim_{n\to\infty}\sqrt[n]{H_n(t)2} \le \lim_{n\to\infty}\sqrt[n]{f_nC^{n+1}M(t_2)} = 4C.$$

because $f_n \sim \dfrac{4^n}{\sqrt{\pi n}}$ and the proof is finished. □

Using results in [DLdeS] we can easily show using Cayley transform that there exists a Krein space realization for the just constructed function $S(\lambda, t_2)$.

Theorem 6.2. *Let $S(\lambda, t_2)$ be an analytic at infinity function with value I there symmetric with respect to the imaginary axis (i.e. satisfies* (2.23)*), then there exists a realization*

$$S(\lambda, t_2) = I - B(t_2)^*\mathbb{X}^{-1}(t_2)(\lambda I - A_1)^{-1}B(t_2)\sigma_1(t_2)$$

such that the Lyapunov equation holds

$$A_1\mathbb{X}(t_2)+\mathbb{X}(t_2)A_1^*+B(t_2)^*\sigma_1(t_2)B(t_2) = 0.$$

Proof: The idea behind this theorem is that one can construct a Krein space realization at the given time t_2^0 and then obtain a vessel with an inner Krein space $\widetilde{\mathcal{K}}$ instead of $\mathcal{H}$, using the formulas in theorem 4.1. Indeed, suppose that there is constructed a Krein space realization at t_2^0 (which will be done in (6.8)):

$$S(\lambda, t_2^0) = I - B_0^* \mathbb{X}_0^{-1}(\lambda I - A_1)^{-1} B_0 \sigma_1(t_2^0).$$

Define next $B(t_2)$ as the solution of (4.1)

$$\frac{d}{dt_2}[B(t_2)\sigma_1(t_2)] + A_1 B(t_2)\sigma_2(t_2) + B(t_2)\gamma(t_2) = 0, \quad B(t_2^0) = B_0$$

and $\mathbb{X}(t_2)$ as a solution of (4.3)

$$\frac{d}{dt_2}\mathbb{X}(t_2) = B(t_2)\sigma_2(t_2)B(t_2)^*, \quad \mathbb{X}(t_2^0) = \mathbb{X}_0.$$

and define $\gamma_*^1(t_2)$ using the Linkage condition (2.15) for which we shall denote by $\Phi_*^1(\lambda, t_2, t_2^0)$ the fundamental matrix of (2.17). Then we will obtain that

$$Y(\lambda, t_2) = \Phi_*^1(\lambda, t_2, t_2^0) S(\lambda, t_2^0) \Phi(\lambda, t_2, t_2^0).$$

On the other hand, in theorem 6.1 we proved that one can construct a function $S(\lambda, t_2)$ with suitably chosen initial values, i.e. with $S(\lambda, t_2^0)$ such that

$$S(\lambda, t_2) = \Phi_*(\lambda, t_2, t_2^0) S(\lambda, t_2^0) \Phi(\lambda, t_2, t_2^0)$$

for the given $\gamma_*(t_2)$. It is remained to notice that

$$Y(\lambda, t_2) S^{-1}(\lambda, t_2) = \Phi_*^1(\lambda, t_2, t_2^0) \Phi_*(\lambda, t_2, t_2^0)$$

is entire and identity at infinity, i.e. is I, which means that

$$Y(\lambda, t_2) = S(\lambda, t_2)$$

and from here immediately follows (as at the second part of 4.1) that $\gamma_*(t_2) = \gamma_*^1(t_2)$ and we have obtained a Krein space realization for the function $S(\lambda, t_2)$ constructed in theorem 6.1.
It remains to show that there always exists a realization at t_2^0. Define a function $Q(\lambda)$ using Cayley transform, which satisfies

$$S(-i\lambda, t_2^0) = (I + \frac{i}{2} Q(\lambda)\sigma_1(t_2^0))(I - \frac{i}{2} Q(\lambda)\sigma_1(t_2^0))^{-1}. \tag{6.5}$$

Actually, this function is given by

$$Q(\lambda) = 2i\sigma_1^{-1}(t_2^0)(I - S(-i\lambda, t_2^0))(I + S(-i\lambda, t_2^0))^{-1}$$

and is well-defined at the neighborhood of infinity with value 0 there. Then from a simple equality, resulting from the symmetry condition (2.23) considered with $-i\lambda$ instead of λ

$$\begin{aligned}(I - S^*(-i\bar{\lambda}, t_2^0))\sigma_1^{-1}(t_2^0)(I + S(-i\lambda, t_2^0)) = \\ = -(I + S^*(-i\bar{\lambda}, t_2^0))\sigma_1^{-1}(t_2^0)(I - S(-i\lambda, t_2^0))\end{aligned}$$

it follows that $Q(\lambda)^* = Q(\bar{\lambda})$ and $Q(\lambda, t_2)$ is zero at the neighborhood of $\lambda = \infty$. Thus [DLdeS, Theorem 3] $Q(\lambda, t_2)$ admits the following Krein space realization

$$Q(\lambda) = \Gamma^+(\widetilde{A} - \lambda I)^{-1}\Gamma$$

where for a Krein space $\widetilde{\mathcal{K}}$, there is a bounded self-adjoint operator $\widetilde{A} \in L(\widetilde{\mathcal{K}})$, and a bounded $\Gamma \in L(\mathcal{E}, \widetilde{\mathcal{K}})$. So, we can insert this realization formula into (6.5) and simplify:

$$\begin{aligned} S(-i\lambda, t_2^0) &= (I + \frac{i}{2}Q(\lambda)\sigma_1(t_2^0))(I - \frac{i}{2}Q(\lambda)\sigma_1(t_2^0))^{-1} = \\ &= (2I - I + \frac{i}{2}Q(\lambda)\sigma_1(t_2^0))(I - \frac{i}{2}Q(\lambda)\sigma_1(t_2^0))^{-1} = \\ &= -I + 2(I - \frac{i}{2}Q(\lambda)\sigma_1(t_2^0))^{-1} = \\ &= -I + 2(I - \frac{i}{2}\Gamma^+(\widetilde{A} - \lambda I)^{-1}\Gamma\sigma_1(t_2^0))^{-1} \end{aligned}$$

There is a simple formula [BGR] for evaluating the inverse of a matrix in a realized form:

$$(I - \frac{i}{2}\Gamma^+(\widetilde{A} - \lambda I)^{-1}\Gamma\sigma_1(t_2^0))^{-1} = I + \frac{i}{2}\Gamma^+(\widetilde{A}^\times - \lambda I)^{-1}\Gamma\sigma_1(t_2^0),$$

where $\widetilde{A}^\times = \widetilde{A} - \frac{i}{2}\Gamma\sigma_1(t_2^0)\Gamma^+$. So, the last formula becomes

$$\begin{aligned} S(-i\lambda, t_2) &= -I + 2(I - i\Gamma^+(\widetilde{A} - \lambda I)^{-1}\Gamma\sigma_1(t_2^0))^{-1} = \\ &= -I + 2(I + \frac{i}{2}\Gamma^+(\widetilde{A}^\times - \lambda I)^{-1}\Gamma\sigma_1(t_2^0)) = \\ &= I + i\Gamma^+(\widetilde{A}^\times - \lambda I)^{-1}\Gamma\sigma_1(t_2^0) = \\ &= I - \Gamma^+(i\widetilde{A}^\times - i\lambda I)^{-1}\Gamma\sigma_1(t_2^0) \end{aligned} \tag{6.6}$$

Let us define $A_1 = -i\widetilde{A}^\times$ then we obtain that

$$\begin{aligned} A_1 + A_1^+ &= -i\widetilde{A}^\times + i(\widetilde{A}^\times)^+ = \\ &= -i(\widetilde{A} - \frac{i}{2}\Gamma\sigma_1(t_2^0)\Gamma^+) + i(\widetilde{A}^+ + \frac{i}{2}\Gamma\sigma_1(t_2^0)\Gamma^+) = \\ &= -\Gamma\sigma_1(t_2^0)\Gamma^+, \end{aligned} \tag{6.7}$$

since $\widetilde{A}$ is selfadjoint. In the Krein space, the Krein space adjoint may be represented by an invertible selfadjoint operator $\widetilde{\mathbb{X}}$ in (the Hilbert

space sense), when one considers the Krein space with its inner product $[\cdot,\cdot]$ as a Hilbert space with its initial norm $\langle\cdot,\cdot\rangle$:

$$[u,v] = \langle \widetilde{\mathbb{X}}u, v\rangle, \forall u,v \in \widetilde{\mathcal{K}}.$$

Then the formulas for Krein space adjoint becomes:

$$A_1^+ = \widetilde{\mathbb{X}}^{-1}A_1^*\widetilde{\mathbb{X}}, \quad \Gamma^+ = \widetilde{\mathbb{X}}\Gamma^*$$

and the last equation (6.7)

$$\begin{array}{l} A_1 + A_1^+ = -\Gamma\sigma_1(t_2^0)\Gamma^+ \Leftrightarrow \\ A_1 + \widetilde{\mathbb{X}}^{-1}A_1^*\widetilde{\mathbb{X}} = \Gamma\sigma_1(t_2^0)\Gamma^*\widetilde{\mathbb{X}} \Leftrightarrow \\ A_1\widetilde{\mathbb{X}}^{-1} + \widetilde{\mathbb{X}}^{-1}A_1^* = \Gamma\sigma_1(t_2^0)\Gamma^* \end{array}$$

which is the Lyapunov equation (3.8) at t_2^0 after defining $B_0 = \Gamma$ and $\mathbb{X}_0 = \widetilde{\mathbb{X}}^{-1}$. Notice that Lyapunov equation (3.8) will also hold for all t_2 as a result of lemma 2.4. Moreover, substituting λ for $-i\lambda$ in (6.6), we also obtain

$$S(\lambda, t_2^0) = I - B_0^*\mathbb{X}_0^{-1}(\lambda I - A_1)^{-1}B_0\sigma_1(t_2^0). \tag{6.8}$$

□

It turns out that there are examples for which there is no a realization with strictly positive $\mathbb{X}$. For, example considering the Non-Linear Schrödinger equation parameters, defined in section 5.6 for *constant* β

$$\gamma_* = \begin{bmatrix} 0 & \beta \\ \beta^* & 0 \end{bmatrix}$$

we can easily find that the fundamental matrices for the input and output LDEs are (for convenience $t_2^0 = 0$)

$$\Phi(\lambda, t_2) = \begin{bmatrix} e^{\frac{\lambda}{2}t_2} & 0 \\ 0 & e^{\frac{-\lambda}{2}t_2} \end{bmatrix},$$

$$\Phi_*(\lambda, t_2) = \begin{bmatrix} \dfrac{-\beta\beta^*}{2\Gamma}[\dfrac{E}{\Gamma - \frac{\lambda}{2}} + \dfrac{E^{-1}}{\Gamma + \frac{\lambda}{2}}] & \dfrac{\beta}{2\Gamma}[E - E^{-1}] \\ \dfrac{-\beta^*}{2\Gamma}[E - E^{-1}] & \dfrac{(\Gamma - \frac{\lambda}{2})E}{2\Gamma} + \dfrac{(\Gamma + \frac{\lambda}{2})E^{-1}}{2\Gamma} \end{bmatrix}$$

where

$$\Gamma = \sqrt{\frac{\lambda^2}{4} - \beta\beta^*}, \quad E = \exp(\Gamma t_2).$$

Let us search for a constant in t_2 matrix $S(\lambda, 0) = \begin{bmatrix} a(\lambda) & b(\lambda) \\ c(\lambda) & d(\lambda) \end{bmatrix}$ such that the constructed $S(\lambda, t_2) = \Phi_*(\lambda, t_2)S(\lambda, 0)\Phi^{-1}(\lambda, t_2)$ satisfies

conditions of the proposition 2.5. Considering the identity at infinity requirement, we come to the conclusion that

$$c(\lambda) = \frac{-\beta^*}{\Gamma + \frac{\lambda}{2}} a(\lambda), \quad b(\lambda) = \frac{-\beta}{\Gamma + \frac{\lambda}{2}} d(\lambda),$$

where the square root at the Right Half Plane (RHP) is taken so that at the infinity its argument coincides with the argument of $\frac{\lambda}{2}$. In this case the expression $\Gamma + \frac{\lambda}{2}$ is analytic in the whole $\mathbb{C}$, except for a cut from $-\sqrt{2}|\beta|$ to $\sqrt{2}|\beta|$. In other words this can not be defined as an analytic function at the right half plane. On the other hand, if it were possible to realize this function with A_1, satisfying Lyapunov equation (3.8) with positive $\mathbb{X}$, this would mean that the spectrum of A_1 is at the left hand side, since $\sigma_1 = I$ and is a positive matrix:

$$A_1\mathbb{X} + \mathbb{X}A_1^* = -B^*B \le 0.$$

As a result $S(\lambda, 0)$ would have spectrum only at the left hand side of the complex plane, contradicting the previous argument.
Finally, we present in this section a lemma, which says that one can always define moments so that the corresponding Pick matrices are positive, but as the example above shows they will not always define an analytic function (the radius of convergence may be zero)

Lemma 6.3. *Suppose that equations* (5.5), (5.4) *hold so that the eigenvalues of the Pick matrix $\mathbb{P}_n$ are strictly positive on a finite interval I. Suppose that initial values are chosen for H_{2n+1}, then there exists an initial values for H_{2n+2} such that all eigenvalues of P_{n+1} are positive on arbitrary finite interval around t_2^0.*

Proof: Let us use Sylvester criterion for positive definiteness of a matrix. Denote the entries of the matrix $(-1)^n H_{2n}\sigma_1^{-1}$ as follows

$$(-1)^n H_{2n}\sigma_1^{-1} = [h_{ij}] = \begin{bmatrix} h_{11} & h_{12} & \dots & h_{1p} \\ h_{21} & h_{22} & \dots & h_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ h_{p1} & h_{p2} & \dots & h_{pp} \end{bmatrix}, \quad D_n = [d_{ij}]$$

and suppose that $\mathbb{P}_n > 0$ on a fixed interval $\mathcal{I}$ including t_2^0. We will consider main minors of the matrix $\mathbb{P}_{n+1}$. First minors come from $\mathbb{P}_n$ and as a result are positive, the next one is

$$\begin{bmatrix} \mathbb{P}_n & C_n^1 \\ (C_n^1)^* & h_{11} + d_{11} \end{bmatrix},$$

where C_n^1 is the first column of the matric C_n. Its determinant, using a formula for block matrix is

$$\det \begin{bmatrix} \mathbb{P}_n & C_n^1 \\ (C_n^1)^* & h_{11}+d_{11} \end{bmatrix} = \det \mathbb{P}_n \det(h_{11}+d_{11}-(C_n^1)^*\mathbb{P}_n^{-1}C_n^1).$$

In order to obtain here a positive result, we have to demand that (since $\det \mathbb{P}_n > 0$ and since $h_{11}+d_{11}$ is real valued)

$$\Re h_{11} > (C_n^1)^*\mathbb{P}_n^{-1}C_n^1 - \Re d_{11}.$$

Using the continuity and taking $h_{11}(t_2^0)$ big enough we obtain that the same inequality will hold on the interval $\mathcal{I}$. Consider now the next main minor

$$\begin{bmatrix} \mathbb{P}_n & C_n^1 & C_n^2 \\ (C_n^1)^* & h_{11}+d_{11} & h_{12}+d_{12} \\ (C_n^2)^* & h_{21}+d_{21} & h_{22}+d_{22} \end{bmatrix}.$$

Its determinant can be expressed as

$$\begin{aligned} &\det \begin{bmatrix} \mathbb{P}_n & C_n^1 & C_n^2 \\ (C_n^1)^* & h_{11}+d_{11} & h_{12}+d_{12} \\ (C_n^2)^* & h_{21}+d_{21} & h_{22}+d_{22} \end{bmatrix} = \\ &= \det \begin{bmatrix} \mathbb{P}_n & C_n^1 \\ (C_n^1)^* & h_{11}+d_{11} \end{bmatrix} \times \\ &\times \det\big(h_{22}+d_{22} - \begin{bmatrix} (C_n^2)^* & h_{21}+d_{21} \end{bmatrix} \begin{bmatrix} \mathbb{P}_n & C_n^1 \\ (C_n^1)^* & h_{11}+d_{11} \end{bmatrix}^{-1} \begin{bmatrix} C_n^2 \\ h_{12}+d_{12} \end{bmatrix}\big) \end{aligned}$$

from where we obtain that the necessary condition for its positiveness is

$$\Re h_{22} > \begin{bmatrix} (C_n^2)^* & h_{21}+d_{21} \end{bmatrix} \begin{bmatrix} \mathbb{P}_n & C_n^1 \\ (C_n^1)^* & h_{11}+d_{11} \end{bmatrix}^{-1} \begin{bmatrix} C_n^2 \\ h_{12}+d_{12} \end{bmatrix} - \Re d_{22}$$

And so on for all the real parts of diagonal elements h_{ii}. It remains to remark that the conditions (5.5), (5.4) do not involve real parts of h_{ii} and they can be determined from the positive definiteness of P_{n+1}. □

7. Generalized Nevanlinna-Pick interpolation problem

We address the two kinds of Nevanlinna-Pick interpolation problems 1.4 and 1.5, defined in the introduction. The first one is solved using the notion of positive pairs and for the second one we present a criterion in terms of the of the initial data.

7.1. **Linear Fractional Transformations in terms of intertwining positive pairs.** Suppose that we are given a data of the NP interpolation problem 1.4. Following the notations of corollary 3.3 let us write

$$B(t_2) = \begin{bmatrix} -\xi_1(t_2)^* S(w_1,t_2)^* & \xi_1(t_2)^* \\ \vdots & \vdots \\ -\xi_n(t_2)^* S(w_n,t_2)^* & \xi_n(t_2)^* \end{bmatrix}, \ A_1 = \operatorname{diag}[-w_1^*, \ldots, -w_n^*].$$

Let us denote by capital Greek letters the following vessel parameters

$$\begin{aligned} \Sigma_1(t_2) &= \begin{bmatrix} -\sigma_1(t_2) & 0 \\ 0 & \sigma_1(t_2) \end{bmatrix} = J, \\ \Sigma_2(t_2) &= \begin{bmatrix} \sigma_2(t_2) & 0 \\ 0 & \sigma_2(t_2) \end{bmatrix}, \\ \Gamma(t_2) &= \begin{bmatrix} \gamma_*(t_2) & 0 \\ 0 & \gamma(t_2) \end{bmatrix}, \end{aligned}$$

then simple calculations show that $B(t_2)$ satisfies (4.1)

$$\frac{d}{dt_2}[B(t_2)\Sigma_1(t_2)] + A_1 B(t_2)\Sigma_2(t_2) + B(t_2)\Gamma(t_2) = 0, \quad B(t_2^0) = B.$$

Suppose that $\mathbb{X}(t_2) > 0$ is a solution of

$$\begin{aligned} A_1\mathbb{X}(t_2) + \mathbb{X}(t_2)A_1^* + B(t_2)\Sigma_1(t_2)B^*(t_2) = 0, \\ \frac{d}{dt_2}\mathbb{X}(t_2) = B(t_2)\Sigma_2(t_2)B^*(t_2), \end{aligned}$$

which is always possible if $\Re w_i \neq 0$ for each $i = 1, \ldots, n$. Then the following collection

$$\mathfrak{V} = \{A_1, B(t_2), \mathbb{X}(t_2); \Sigma_1(t_2), \Sigma_2(t_2), \Gamma(t_2), \Gamma_*(t_2); \mathbb{C}^{2n}; \mathcal{E} \oplus \mathcal{E}\}.$$

is a vessel for $\Gamma_*(t_2)$ defined from the linkage condition (2.15)

$$\begin{aligned} \Gamma_*(t_2) = \Gamma(t_2) + \Sigma_1(t_2)B(t_2)^*\mathbb{X}^{-1}(t_2)B(t_2)\Sigma_2(t_2) - \\ -\Sigma_2(t_2)B(t_2)^*\mathbb{X}^{-1}(t_2)B(t_2)\Sigma_1(t_2). \end{aligned}$$

Transfer function of the vessel $\mathfrak{V}$ is

$$\Theta(\lambda, t_2) = I_{2p} - B(t_2)^*\mathbb{X}^{-1}(t_2)(\lambda I_n - A_1)^{-1}B(t_2)\Sigma_1(t_2),$$

which is in $\boldsymbol{\mathcal{SI}}(\mathbf{U}_* \oplus \mathbf{U}, \widetilde{\mathbf{U}})$ for

$$\widetilde{\mathbf{U}} = \lambda\Sigma_2(t_2) - \Sigma_1(t_2)\frac{d}{dt_2} + \Gamma_*(t_2).$$

If we denote further the decomposition of $\Theta(t_2)$ as

$$\Theta(\lambda,t_2)=\begin{bmatrix}\Theta_{11}(\lambda,t_2) & \Theta_{12}(\lambda,t_2)\\ \Theta_{21}(\lambda,t_2) & \Theta_{22}(\lambda,t_2)\end{bmatrix}$$

then if one defines $S_0(\lambda,t_2)$ so that

$$(\Theta_{11}+S_0\Theta_{21})^{-1}(\Theta_{12}+S_0\Theta_{22})=S,$$

then the function $S(\lambda,t_2)$ usually does not intertwine solutions of LDEs with spectral parameter λ.
Instead, we define

$$W(\lambda,t_2)=\begin{bmatrix}W_1(\lambda,t_2) & W_2(\lambda,t_2)\end{bmatrix}$$

so that

$$W_1\Theta_{11}+W_2\Theta_{21}=I_p,\quad W_1\Theta_{12}+W_2\Theta_{22}=S,$$

then the following lemma holds

Lemma 7.1. *The pair of functions $W(\lambda,t_2)$ is in $\mathcal{SI}(\widetilde{\mathbf{U}},\mathbf{U}_*)$ and $\dfrac{W(\lambda,t_2)^*\Sigma_1(t_2)W(\mu,t_2)}{\bar\lambda+\mu}\geq 0$ on the domain of analyticity of $W(\lambda,t_2)$.*

Proof: Since $\Theta(\lambda,t_2)$ is invertible for all λ out of the spectrum of A_1, an element of $\widetilde{U}$ is of the form $\Theta(\lambda,t_2)\begin{bmatrix}y_\lambda(t_2)\\ u_\lambda(t_2)\end{bmatrix}$, where $y_\lambda(t_2)$, $u_\lambda(t_2)$ satisfy (2.17) and (2.16) respectively. Then

$$\begin{aligned}W(\lambda,t_2)\Theta(\lambda,t_2)\begin{bmatrix}y_\lambda(t_2)\\ u_\lambda(t_2)\end{bmatrix}=\\ =\begin{bmatrix}W_1(\lambda,t_2) & W_2(\lambda,t_2)\end{bmatrix}\begin{bmatrix}\Theta_{11}(\lambda,t_2) & \Theta_{12}(\lambda,t_2)\\ \Theta_{21}(\lambda,t_2) & \Theta_{22}(\lambda,t_2)\end{bmatrix}\begin{bmatrix}y_\lambda(t_2)\\ u_\lambda(t_2)\end{bmatrix}=\\ =\begin{bmatrix}W_1\Theta_{11}+W_2\Theta_{21} & W_1\Theta_{12}+W_2\Theta_{22}\end{bmatrix}\begin{bmatrix}y_\lambda(t_2)\\ u_\lambda(t_2)\end{bmatrix}=\\ =\begin{bmatrix}I & S(\lambda,t_2)\end{bmatrix}\begin{bmatrix}y_\lambda(t_2)\\ u_\lambda(t_2)\end{bmatrix}=y_\lambda(t_2)+S(\lambda,t_2)u_\lambda(t_2)\in\mathbf{U}_*,\end{aligned}$$

since $y_\lambda(t_2)$ and $S(\lambda,t_2)u_\lambda(t_2)$ are in $\mathbf{U}_*$.
From the formula $W\Theta=\begin{bmatrix}I & S\end{bmatrix}$, it follows that

$$W(\lambda,t_2)=\begin{bmatrix}I & S(\lambda,t_2)\end{bmatrix}\Theta^{-1}(\lambda,t_2).$$

Consequently, the expression $\dfrac{W(\lambda,t_2)^*\Sigma_1(t_2)W(\mu,t_2)}{\bar\lambda+\mu}$ considered on the domain of analyticity of $W(\cdot,t_2)$ becomes

$$\frac{W(\lambda,t_2)\Sigma_1(t_2)W^*(\mu,t_2)}{\lambda+\bar\mu} =$$
$$= \begin{bmatrix} I & S(\lambda,t_2) \end{bmatrix} \frac{\Theta^{-1}(\lambda,t_2)\Sigma_1(t_2)\Theta^{-1*}(\mu,t_2)}{\lambda+\bar\mu} \begin{bmatrix} I \\ S^*(\mu,t_2) \end{bmatrix}.$$

Since $\Theta(\lambda,t_2)$ is a transfer function of a conservative vessel, its inverse is a transfer function too and satisfies

$$\frac{\Theta^{-1}(\lambda,t_2)\Sigma_1(t_2)\Theta(\mu,t_2)^{-1*}-\Sigma_1(t_2)}{\lambda+\bar\mu} \geq 0$$

and consequently,

$$\begin{aligned}\frac{W(\lambda,t_2)\Sigma_1(t_2)W^*(\mu,t_2)}{\lambda+\bar\mu} &\geq \frac{1}{\lambda+\bar\mu}\begin{bmatrix} I & S(\lambda,t_2) \end{bmatrix}\Sigma_1(t_2)\begin{bmatrix} I \\ S^*(\mu,t_2) \end{bmatrix}\\ &\geq \frac{1}{\lambda+\bar\mu}\begin{bmatrix} I & S(\lambda,t_2) \end{bmatrix}\begin{bmatrix} -\sigma_1(t_2) & 0 \\ 0 & \sigma_1(t_2) \end{bmatrix}\begin{bmatrix} I \\ S^*(\mu,t_2) \end{bmatrix}\\ &\geq \frac{S(\lambda,t_2)\sigma_1(t_2)S^*(\mu,t_2)-\sigma_1(t_2)}{\lambda+\bar\mu}\\ &\geq 0\end{aligned}$$

by the properties of transfer functions for vessels. □

As a consequence of this theorem, we define

Definition 7.2. *A pair of functions*

$$W(\lambda,t_2) = \begin{bmatrix} W_1(\lambda,t_2) & W_2(\lambda,t_2) \end{bmatrix}$$

*such that the matrix $W(\lambda,t_2)$ has full rank is called **positive** if the conditions of lemma 7.1 hold:*

$$W(\lambda,t_2)\in \boldsymbol{\mathcal{SI}}(\widetilde{\mathbf{U}},\mathbf{U}_*), \quad W(\lambda,t_2)JW(\lambda,t_2)\geq 0 \text{ on } \mathbb{C}_+$$

Previous arguments result in the following "partial solution" of the Nevanlinna-Pick interpolation problem 1.4:

Corollary 7.3. *There exists a one-to-one correspondence between solution of the Nevanlinna-Pick interpolation problem 1.4 and the positive pairs $W(\lambda,t_2)$, which are analytic at the interpolation points.*

7.2. **Nevanlinna-Pick interpolation problem 1.5.** Using the previous results, we present a criterion for solvability of the Problem 1.5.

Theorem 7.4. *Given $\mathbb{C}^{p\times p}$-valued functions σ_1, σ_2, γ, an interval* I *and N quadruples $< t_2^j, w_j, \xi_j, \eta_j >$ where $t_2^j\in$ I, $w_j\in\mathbb{C}_+$, $\xi_j,\eta_j\in\mathbb{C}^{1\times p}$ $j=1,\dots N$, and assume that the corresponding matrices $\widetilde{\mathbb{X}}_i>0$. Then*

there exists a solution of the Nevanlinna-Pick problem 1.5, i.e. there exists a function $S \in \mathcal{RSI}$ satisfying $S(w_i, t_2^i)\xi_i = \eta_i$ if and only if there exists $n \in \mathbb{N}$, matrices $A_0^i, \mathbb{X}_0^i, \in \mathbb{C}^{(n-1)\times(n-1)}$ with $\mathbb{X}_0^i > 0$, $B_0^i \in \mathbb{C}^{p\times(n-1)}, V_{ij} \in \mathbb{C}^{n\times n}$ such that for $A_i, B_i, \mathbb{X}_i$ defined by

$$(7.1) \quad B_i = \begin{bmatrix} B_0^i \\ \eta_i - \xi_i \end{bmatrix}$$

$$(7.2) \quad \mathbb{X}_i = \begin{bmatrix} \mathbb{X}_0^i & 0 \\ 0 & \widetilde{\mathbb{X}}_i \end{bmatrix}$$

$$(7.3) \quad A_i = \begin{bmatrix} A_0^i & \dfrac{B_0^i\sigma_1(t_2^i)\xi_i^*}{\widetilde{\mathbb{X}}_i} \\ -\eta_i\sigma_1(t_2^i)(B_0^i)^*(\mathbb{X}_0^i)^{-1} & -w_i^* - \dfrac{\eta_i\sigma_1(t_2^i)(\eta_i^* - \xi_i^*)}{\widetilde{\mathbb{X}}_i} \end{bmatrix}$$

it holds that

(1) $A_i = V_{ij}A_jV_{ij}^{-1}$,

(2) $\oint(\lambda I - A_i)^{-1}B_i\sigma_1^{-1}(t_2^j)\Phi^{-1}(\lambda, t_2^j, t_2^i)d\lambda = V_{ij}B_j$.

and the matrix $X(t_2)$,

$$\mathbb{X}(t_2) = \mathbb{X}_i + \int_{t_2^i}^{t_2} B_i(y)\sigma_2(y)(B_i(y))^* dy$$

is invertible on the interval I.

Proof: For each t_2^i all the functions which satisfy $S(w_i, t_2^i)\xi_i = \eta_i$ are of the form $T_{\Theta_i}(S_0^i(\lambda, t_2))$, provided $\widetilde{\mathbb{X}}_i = \dfrac{\xi_i\sigma_1(t_2^i)\xi_i^* - \eta_i\sigma_1(t_2^i)\eta_i^*}{w_i^* + w_i} > 0$. Here

$$\Theta_i(\lambda) = \begin{bmatrix} I + \dfrac{\eta_i^*\eta_i\sigma_1}{\widetilde{\mathbb{X}}_i(\lambda + w_i^*)} & \dfrac{\eta_i^*\xi_i\sigma_1}{\widetilde{\mathbb{X}}_i(\lambda + w_i^*)} \\ -\dfrac{\xi_i^*\eta_i\sigma_1}{\widetilde{\mathbb{X}}_i(\lambda + w_i^*)} & I - \dfrac{\xi^*\xi\sigma_1}{\mathbb{X}(\lambda + w^*)} \end{bmatrix}$$

and $S_0^i \in \mathcal{RS}$. Given a minimal realization

$$S_0^i(\lambda) = I - (B_0^i)^*(\mathbb{X}_0^i)^{-1}(\lambda I - A_0^i)^{-1}B_0^i\sigma_1(t_2^i)$$

of S_0^i, we shall obtain from formulas (3.15), (3.16), (3.17) the formulas (7.1), (7.2), (7.3) in the theorem. In view of Theorem 4.8, a necessary and sufficient condition to obtain the same function $S(\lambda, t_2)$ for every i is that there exist invertible constant matrices V_{ij} such that the operators A_i are similar. In other words there must exist V_{ij} such that $A_i = V_{ij}A_jV_{ij}^{-1}$. Moreover, the second part of theorem 4.8 tells that

additionally the equality

$$\oint (\lambda I - A_i)^{-1} B_i \sigma_1^{-1}(t_2^j)\Phi^{-1}(\lambda, t_2^j, t_2^i) d\lambda = V_{ij} B_j$$

must hold. $\square$

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